Vertex-coloring edge-weightings of graphs

Gerard J. Chang[123*] Changhong Lu[4,5†] Jiaojiao Wu[1‡] Qinglin Yu[6§]

[1]Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan
[2]Institute for Mathematical Sciences, National Taiwan University, Taipei 10617, Taiwan
[3]National Center for Theoretical Sciences, Taipei, Taiwan
[4]Department of Mathematics, East China Normal University, Shanghai 200062, P. R. China
[5]Institute of Theoretical Computing, ECNU, Shanghai 200062, P. R. China
[6]Department of Mathematics and Statistics, Thompson Rivers University, Kamloops, Canada

Abstract

A k-*edge-weighting* of a graph G is a mapping $w : E(G) \to \{1, 2, \ldots, k\}$. An edge-weighting w induces a vertex coloring $f_w : V(G) \to \mathbb{N}$ defined by $f_w(v) = \sum_{v \in e} w(e)$. An edge-weighting w is *vertex-coloring* if $f_w(u) \neq f_w(v)$ for any edge uv. The current paper studies the parameter $\mu(G)$, which is the minimum k for which G has a vertex-coloring k-edge-weighting. Exact values of $\mu(G)$ are determined for several classes of graphs, including trees and r-regular bipartite graph with $r \geq 3$.

Keywords. Edge-weighting; vertex-coloring; tree; bipartite graph.

1 Introduction

A k-*edge-weighting* of a graph G is a mapping $w : E(G) \to \{1, 2, \ldots, k\}$. An edge-weighting w induces a vertex coloring $f_w : V(G) \to \mathbb{N}$ defined by $f_w(v) = \sum_{v \in e} w(e)$. An edge-weighting w is *vertex-coloring* (respectively, *vertex-injective*) if $f_w(u) \neq f_w(v)$ for any edge uv (respectively, every pair of distinct vertices u and v). Denote by $\mu(G)$ (respectively, $\mu^*(G)$) the minimum k for which G has a vertex-coloring (respectively, vertex-injective) k-edge-weighting. We refer a graph *non-trivial* if it contains no single edge as a component. Notice that $\mu(G) \leq \mu^*(G)$ for every non-trivial graph G.

An edge-weighting is *adjacent vertex-distinguishing* (respectively, *vertex-distinguishing*) if for any edge uv (respectively, every pair of distinct vertices u and v), the multi-set of

*E-mail: gjchang@math.ntu.edu.tw.
†E-mail: chlu@math.ecnu.edu.cn.
‡E-mail: wujj0007@yahoo.com.tw.
§E-mail: yu@tru.ca.

weights appearing on edges incident to u is distinct from the multi-set of weights appearing on the edges incident to v. Denote by $\mu_m(G)$ (respectively, $\mu_m^*(G)$) the minimum k for which G has an adjacent vertex-distinguishing (respectively, vertex-distinguishing) k-edge-weighting. Notice that $\mu_m(G) \leq \mu_m^*(G)$ for every non-trivial graph G. Then, upper bounds for $\mu(G)$ (respectively, $\mu^*(G)$) provide upper bounds for $\mu_m(G)$ (respectively, $\mu_m^*(G)$).

It is clear that a vertex-coloring (respectively, vertex-injective) edge-weighting is adjacent vertex-distinguishing (respectively, vertex-distinguishing), but the converse is not necessarily true. Consequently, $\mu_m(G) \leq \mu(G)$ and $\mu_m^*(G) \leq \mu^*(G)$ for every non-trivial graph G.

Adjacent vertex-distinguishing edge-weighting and vertex-distinguishing edge-weighting have been studied by many researchers [4, 6, 5, 7]. Karoński, Luczak and Thomason [10] proved that $\mu_m(G) \leq 213$ for every non-trivial graph and that $\mu_m(G) \leq 30$ for every graph with minimum degree at least 10^{99}. Addario-Berry et al. [1] improved the results to $\mu_m(G) \leq 4$ for every non-trivial graph and $\mu_m(G) \leq 3$ for every graph of minimum degree at least 1000.

For vertex-coloring edge-weighting, Karoński, Luczak and Thomason [10] posed the following question:

Question. Does $\mu(G) \leq 3$ for every non-trivial graph G?

Karoński, Luczak and Thomason [10] showed that if G is a k-colorable graph with k odd then G admits a vertex-coloring k-edge-weighting. So, for the class of 3-colorable graphs, including bipartite graphs, the answer is affirmative. However, in general, this question is still open. The first constant bound was obtained by Addario-Berry et al. [2], who showed that $\mu(G) \leq 30$ for every non-trivial graph G. The bound is improved to $\mu(G) \leq 16$ in [3], to $\mu(G) \leq 13$ in [11], and to $\mu(G) \leq 5$ in [9].

Even we are still far from providing a positive answer to the question, actually $\mu(G) \leq 2$ for many graphs (in fact, experiments suggest (see [10]) that $\mu(G) \leq 2$ for almost all graphs). The current paper is devoted to study graphs with such a property. We determine $\mu(G)$ for some classes of graphs with this property, including trees and r-regular bipartite graphs with $r \geq 3$.

In the rest of this section, we fix some notation. For $n \geq 1$, the *n-path* P_n is the graph with vertex set $\{v_i : 1 \leq i \leq n\}$ and edge set $\{v_iv_{i+1} : 1 \leq i \leq n-1\}$. For $n \geq 3$, the *n-cycle* C_n is the graph with vertex set $\{v_i : 1 \leq i \leq n\}$ and edge set $\{v_iv_{i+1} : 1 \leq i \leq n\}$, where $v_{n+1} = v_1$. The *complete graph* K_n is the graph with vertex set $\{v_i : 1 \leq i \leq n\}$ and edge set $\{v_iv_j : 1 \leq i < j \leq n\}$. The *complete bipartite graph* $K_{m,n}$ is the graph with vertex set $\{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ and edge set $\{u_iv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. The *neighborhood* of a vertex v is the set $N(v) = \{u : uv \in E(G)\}$, and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d(v) = |N(v)|$. We use $\delta(G)$ to denote the minimum degree of a vertex in a graph G.

2 $\mu(G)$ for some classes of graphs

This section establishes values of $\mu(G)$ for some classes of graphs, including paths, cycles, complete graphs and complete bipartite graphs.

Fact 1 *For every non-trivial graph G, $\mu(G) = 1$ if and only if G has no adjacent vertices with the same degree.*

Fact 2 $\mu(P_3) = 1$ *and* $\mu(P_n) = 2$ *for* $n \geq 4$.

Proof. This follows from Fact 1 and the fact that the following mapping w is a vertex-coloring 2-edge-weighting: $w(v_iv_{i+1}) = 1$ for $i \equiv 1, 2 \pmod 4$ and $w(v_iv_{i+1}) = 2$ for $i \equiv 3, 4 \pmod 4$. ∎

Proposition 3 $\mu(C_n) = 2$ *for* $n \equiv 0 \pmod 4$ *and* $\mu(C_n) = 3$ *for* $n \not\equiv 0 \pmod 4$.

Proof. First, $\mu(C_n) \geq 2$ by Fact 1. For the case when $n \equiv 0 \pmod 4$, $\mu(C_n) = 2$ follows from that the following mapping w is a vertex-coloring 2-edge-weighting: $w(v_iv_{i+1}) = 1$ for $i \equiv 1, 2 \pmod 4$ and $w(v_iv_{i+1}) = 2$ for $i \equiv 3, 4 \pmod 4$.

For the case $n = 4k+r$, $1 \leq r \leq 3$, $\mu(C_n) \leq 3$ follows from that the following mapping w is a vertex-coloring 3-edge-weighting: $w(v_iv_{i+1}) = 1$ for $i \equiv 1, 2 \pmod 4$ and $w(v_iv_{i+1}) = 2$ for $i \equiv 3, 4 \pmod 4$ with the modifications that $w(v_{4k+1}v_{4k+2}) = w(v_{4k+2}v_{4k+3}) = 3$ and

$w(v_{4k+3}v_{4k+4}) = 2$. On the other hand, we claim that $\mu(C_n) \neq 2$. Suppose to the contrary that C_n has a vertex-coloring 2-edge-weighting w. Then, $f_w(v_{i+1}) \neq f_w(v_{i+2})$ implies $w(v_iv_{i+1}) \neq w(v_{i+2}v_{i+3})$ and so $w(v_iv_{i+1}) = w(v_{i+4}v_{i+5})$, where the indices are taken modulo 4. These in turn imply that $w(v_iv_{i+1}) \neq w(v_{i+4j+2}v_{i+4j+3})$. This is a contradiction since $v_i = v_{i+n} = v_{i+4j+2}$ when $r = 2$ with $j = \frac{n-2}{4}$ and $v_i = v_{i+2n} = v_{i+4j+2}$ when $r = 1, 3$ with $j = \frac{n-1}{2}$. ∎

Proposition 4 *If $n \geq 3$, then $\mu(K_n) = 3$.*

Proof. We first consider the following mapping w: $w(v_iv_j) = 1$ for $i + j \leq n$, $w(v_iv_n) = 3$ for $\lfloor \frac{n+2}{2} \rfloor \leq i \leq n-1$, and $w(v_iv_j) = 2$ for all other edges. It is straightforward to check that $f_w(v_i) = n - 1 + i$ for $1 \leq i \leq n-1$ and $f_w(v_n) = \lfloor \frac{5n-5}{2} \rfloor$. Hence, f_w is vertex-coloring and so $\mu(K_n) \leq 3$.

On the other hand, we claim that $\mu(K_n) \neq 2$. Suppose to the contrary that K_n has a vertex-coloring 2-edge-weighting w. Then, each $f_w(v_i)$ is one of the n possible values in $\{n-1, n, \ldots, 2n-2\}$. So, there is exactly one v_i (resp. v_j) with $f_w(v_i) = n-1$ (resp. $f_w(v_j) = 2n-2$). The first equation implies that $w(v_iv_j) = 1$ while the second one implies that $v(v_jv_i) = 2$, a contradiction. Thus, $\mu(K_n) = 3$. ∎

Proposition 5 *$\mu(K_{m,n}) = 1$ when $m \neq n$ and $\mu(K_{m,n}) = 2$ when $m = n \geq 2$.*

Proof. The former case follows from Fact 1. The latter case follows from that for $m = n \geq 2$ the following mapping w is a vertex-coloring 2-edge-weighting: $w(u_iv_j) = 1$ and $w(u_mv_j) = 2$ for $1 \leq i \leq m-1$ and $1 \leq j \leq n$. ∎

The *theta graph* $\theta(\ell_1, \ell_2, \ldots, \ell_r)$ is the graph obtained from r disjoint paths of lengths $\ell_1, \ell_2, \ldots, \ell_r$, respectively, by identifying their end-vertices called the *roots* of the graph. Notice that $\theta(\ell_1) = P_{1+\ell_1}$ and $\theta(\ell_1, \ell_2) = C_{\ell_1+\ell_2}$. In the following we only consider the case $r \geq 3$ and assume that $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_r$.

Proposition 6 *Let $G = \theta(\ell_1, \ell_2, \ldots, \ell_r)$ with $r \geq 3$. Then $\mu(G) = 1$ when $\ell_i = 2$ for all i; $\mu(G) = 3$ when $\ell_1 = 1$ and $\ell_i \equiv 1 \pmod 4$ for all $i \neq 1$; and $\mu(G) = 2$ otherwise.*

Proof. The first equality follows from Proposition 1 and that any two adjacent vertices have different degrees if and only if all $\ell_i = 2$.

For the case when $\ell_1 = 1$ with all $\ell_i \equiv 1 \pmod 4$, we claim that $\mu(G) \geq 3$. Suppose, to the contrary that the graph admits a vertex-coloring 2-edge-weighting w. Then, in each path the kth edge must have the different weight from the $(k+2)$th edge, and has the same weight with the $(k+4)$th edge. Consequently, the first edge has the same weight with the last edge in each path of the theta graph. Then, $f_w(u) = f_w(v)$ for the two roots u and v, however, this is impossible as they are adjacent. On the other hand, the following mapping w is a vertex-coloring 3-edge-weighting: for each path of the theta graph, assign the weights $1, 1, 2, 2$ periodically except the last edge assigned with 3.

For the remaining case, we may construct a vertex-coloring 2-edge-weighting as follows. Notice that for a periodical weight assignment $\ldots 1, 1, 2, 2 \ldots$ of a path with first edge e and last edge e', we may properly choose the starting weight such that $w(e) = w(e') = 2$ except for the case when $\ell_i \equiv 3 \pmod 4$ (one of $w(e)$ and $w(e')$ is 1 and the other is 2). We then may properly arrange the weights on edges to make a vertex-coloring 2-edge-weighting even when $\ell_1 = 1$. ∎

3 $\mu(G)$ for bipartite graphs

In this section, we consider $\mu(G)$ for a bipartite graph G. We use $G = (A, B, E)$ to denote a bipartite graph with vertex bipartition (A, B), and edge set E.

Theorem 7 *Every non-trivial connected bipartite graph $G = (A, B, E)$ with $|A|$ even admits a vertex-coloring 2-edge-weighting w such that $f_w(u)$ is odd for $u \in A$ and $f_w(v)$ is even for $v \in B$. Consequently, $\mu(G) \leq 2$.*

Proof. Assume that $A = \{a_1, a_2, \ldots, a_{2r}\}$. Let P_i be a path from a_i to a_{r+i} for $1 \leq i \leq r$. For each edge e, denote $k(e)$ the number of such paths containing e; and for each vertex u, denote $m(u)$ the sum of $k(e)$ of all edges e incident to u. Then $m(u)$ is odd for $u \in A$ and $m(v)$ is even for $v \in B$. Now, let $w(e) = 1$ for any edge e with $k(e)$ odd and $w(e) = 2$ for any edge e with $k(e)$ even. Since $w(e)$ has the same parity as $k(e)$ for each edge e, the color

$f_w(u)$ of a vertex u satisfies $f_w(u) \equiv m(u) \pmod 2$ for $u \in A \cup B$. Consequently, $f_w(u)$ is odd for $u \in A$ and $f_w(v)$ is even for $v \in B$. Hence, w is a vertex-coloring 2-edge-weighting of G. ∎

Theorem 8 *$\mu(G) \leq 2$ for every non-trivial connected bipartite graph $G = (A, B, E)$ with $\delta(G) = 1$.*

Proof. By Theorem 7, we may assume that both of $|A|$ and $|B|$ are odd. Without loss of generality, assume that $d(x) = 1$ for some vertex x in A, and that x is adjacent to a vertex y in B. Then $G - x = (A \setminus \{x\}, B, E \setminus \{xy\})$ is a non-trivial connected bipartite graph with $|A - \{x\}|$ even. By Theorem 7, $G - x$ has a 2-edge-weighting w' so that $f_{w'}(u)$ is odd for $u \in A \setminus \{x\}$ and $f_{w'}(v)$ is even for $v \in B$. Now, extend w' to w for G by assigning $w(xy) = 2$. This gives a vertex-coloring 2-edge-weighting with $f_w(x) = 2$, $f_w(u)$ odd for $u \in A \setminus \{x\}$, $f_w(v)$ even for $v \in B$ and $f_w(y) > 2$. ∎

Corollary 9 *If T is a tree of at least three vertices, then $\mu(T) \leq 2$.*

Theorem 10 *$\mu(G) \leq 2$ for every non-trivial connected bipartite graph $G = (A, B, E)$ if $\lfloor d(u)/2 \rfloor + 1 \neq d(v)$ for any edge $uv \in E(G)$.*

Proof. By Theorem 7, we may assume that both of $|A|$ and $|B|$ are odd. We need a claim first.

Claim. There exists a vertex x, say $x \in B$, such that the vertices of $G - N[x]$ in A are all in a same component of $G - N[x]$.

Choose a vertex x such that the size of a *maximum* component of $G - N[x]$ becomes as large as possible. Without loss of generality, we assume that $x \in B$. Suppose that besides a maximum component $G_1 = (A_1, B_1, E_1)$ the graph $G - N[x]$ has another component $G_2 = (A_2, B_2, E_2)$, where A_1 and A_2 are nonempty subsets of A. Choose $x' \in A_2$. Since G is connected, $N(x)$ has a vertex adjacent to a vertex in B_1. Then, G_1 together with $N[x]$ are in a same component of $G - N[x']$, and then the size of a maximum component of $G - N[x']$ is larger than that of x, a contradiction to the choice of x.

From the claim, we see that $G - N[x]$ has a component $G_1 = (A_1, B_1, E_1)$ with $A_1 = A \setminus N(x)$ and all other components are isolated vertices in B. Now we consider two cases.

Case 1. $d(x)$ is odd. In this case, $|A_1|$ is even. According to Theorem 7, G_1 has a 2-edge-weighting w' such that $f_{w'}(u)$ is odd for $u \in A_1$ and $f_{w'}(v)$ is even for $v \in B_1$. We then extend w' to w for G by assigning the edges incident to x with weight 1 and the remaining edges with weight 2. Then, $f_w(u)$ is odd for $u \in A$ and $f_w(v)$ is even for $v \in B \setminus \{x\}$. Notice that $f_w(x) = d(x)$ and $f_w(u) = 2d(u) - 1$ for all $u \in N(x)$. These imply $f_w(x) \neq f_w(u)$ by hypothesis. Therefore, w is a vertex-coloring 2-edge-weighting of G.

Case 2. $d(x)$ is even. In this case, $|A_1|$ is odd. Notice that there is a vertex $u^* \in N(x)$ adjacent to some vertex $v^* \in B_1$. Let G' be the graph obtained from G_1 by adding the vertex u^* and the edge u^*v^*. According to Theorem 7, G' has a 2-edge-weighting w' so that $f_{w'}(u)$ is odd for $u \in A_1 \cup \{u^*\}$ and $f_{w'}(v)$ is even for $v \in B_1$. We may extend w' to w for G by assigning the edges incident to x, except xu^*, with weight 1 and the remaining edges with weight 2. Then, $f_w(u)$ is odd for $u \in A$ and $f_w(v)$ is even for $v \in B$ except x. Notice that $f_w(x) = 2\lfloor d(x)/2 \rfloor + 1$ for all $u \in N(x) - u^*$. Therefore, w is a vertex-coloring 2-edge-weighting of G. ∎

Consequently, we have the following result which is in fact our first thought.

Corollary 11 $\mu(G) = 2$ *for every r-regular bipartite graph G with $r \geq 3$.*

Notice that the theta graph $G = \theta(\ell_1, \ell_2, \ldots, \ell_r)$ with $\ell_1 = 1$ and all $\ell_i \equiv 1 \pmod 4$ is a bipartite graph with $\mu(G) = 3$.

We conclude the paper by posing the following problem.

Problem. Characterize bipartite graphs with vertex-coloring 2-edge-weighting.

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