On Strong Product of Factor-Critical Graphs

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Abstract

Strong product $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 has vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent whenever $u_1 = u_2$ and v_1 is adjacent to v_2, or u_1 is adjacent to u_2 and $v_1 = v_2$, or u_1 is adjacent to u_2 and v_1 is adjacent to v_2. We investigate the factor-criticality of $G_1 \boxtimes G_2$ and obtain the following:

Let G_1 and G_2 be connected m-factor-critical and n-factor-critical graphs, respectively. Then

(i) if $m \geqslant 0$, $n = 0$, $|V(G_1)| \geqslant 2m + 2$ and $|V(G_2)| \geqslant 4$, then $G_1 \boxtimes G_2$ is $(2m + 2)$-factor-critical;

(ii) if $n = 1$, $|V(G_1)| \geqslant 2m + 3$ and either $m \geqslant 3$ or $|V(G_2)| \geqslant 5$, then $G_1 \boxtimes G_2$ is $(2m + 4 - \varepsilon)$-factor-critical, where $\varepsilon = 0$ if m is even, otherwise $\varepsilon = 1$;

(iii) if $m + 2 \leqslant |V(G_1)| \leqslant 2m + 2$, or $n + 2 \leqslant |V(G_2)| \leqslant 2n + 2$, then $G_1 \boxtimes G_2$ is mn-factor-critical;

(iv) if $|V(G_1)| \geqslant 2m+3$ and $|V(G_2)| \geqslant 2n+3$, then $G_1 \boxtimes G_2$ is $(mn - \min\{[\frac{3m}{2}]_2, [\frac{3n}{2}]_2\})$-factor-critical.

Keywords: Factor-criticality, product graph, strong product, perfect matching, T-join.

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1 Introduction and Notation

The graphs considered in this paper will be finite, undirected, simple and connected. Let G be a graph with vertex set $V(G)$ and m be an integer such that $0 \leqslant m \leqslant |V(G)| - 2$. A graph G is *m-factor-critical* (hereafter 'm-fc') if

- (i) $|V(G)| \equiv m \pmod 2$;
- (ii) for any $S \subseteq V(G)$, if $|S| = m$, then $G - S$ has a perfect matching (i.e., a 1-factor).

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In particular, a graph G is said to be *factor-critical* if $G-u$ has a 1-factor for every $u \in V(G)$ and to be *bicritical* if for every pair of distinct vertices u and v, $G-\{u,v\}$ has a 1-factor. The factor-critical graphs are used as essential "building blocks" for the so-called Gallai-Edmonds matching structure of general graphs and bicritical graphs are studied by Lovász to develop brick-decomposition as powerful tool to determine the dimension of matching lattice (see [7]). A graph G is called *m-extendable* if every matching of size m can be extended to a perfect matching of G. Clearly, a $2m$-fc graph is m-extendable.

Favaron [3] and Yu [9] introduced the concept of m-fc and studied the basic properties of m-fc graphs, independently. Several properties of m-fc graphs will be used in our proofs, so we summarize them as follows.

Theorem 1.1 ([3], [9]) *Let G be an m-fc graph with $m \geqslant 1$. Then*

(a) *G is also $(m-2)$-fc, if $m \geqslant 2$;*

(b) *G is m-connected;*

(c) *G is $(m+1)$-edge-connected. In particular, $\delta(G) \geqslant m+1$.*

Let $c_o(G)$ denote the number of odd components of G. Favaron [3] and Yu [9] also gave a sufficient and necessary condition on m-fc graphs, independently.

Theorem 1.2 ([3], [9]) *A graph G is m-fc if and only if $c_o(G-S) \leqslant |S|-m$, for all $S \subseteq V(G)$ and $|S| \geqslant m$.*

It is natural to study the factor criticality and matching extendability of different types of graph products, since such products contain a large number of 1-factors and they often form a 'skeleton' of Cayley graphs. Some interesting properties of product graphs can be found in [4] and [5]. Here, we investigate the factor-criticality of the strong product of an m-fc and an n-fc graphs.

Strong product $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 has vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1 = u_2$ and v_1 is adjacent to v_2, or u_1 is adjacent to u_2 and $v_1 = v_2$, or u_1 is adjacent to u_2 and v_1 is adjacent to v_2. For example, $K_2 \boxtimes K_2 = K_4$.

The "projection" subgraph of $G_1 \boxtimes G_2$ induced by the vertex set $\{(u, v_0) \mid u \in V(G_1), v_0 \in V(G_2)\}$ will be denoted by $G_1^{v_0}$. It is called a *row* of $G_1 \boxtimes G_2$. $G_1^{V_0}$ denotes the subgraph of $G_1 \boxtimes G_2$ induced by the vertex set $\{(u, v) \mid u \in V(G_1), v \in V_0 \subseteq V(G_2)\}$. Similarly, we define the notation $G_2^{u_0}$ (a *column* of $G_1 \boxtimes G_2$) and $G_2^{U_0}$. Clearly, $G_1^v \cong G_1$ and $G_2^u \cong G_2$.

One of the important technique throughout the proof is T-join. Let $T \subseteq V(G)$ with $|T|$ even. Let H be a spanning subgraph of G and $d_H(x)$ denote the degree of x in H. Then H is called a *T-join*, if

$$d_H(x) \equiv \begin{cases} 1 \pmod 2, & \text{if } x \in T \\ 0 \pmod 2, & \text{if } x \in V(G) - T. \end{cases}$$

Note that for a T-join H, any vertex of T is of odd degree in H and other vertices are of even degree in H. Given a connected graph G and a subset $T \subseteq V(G)$ with $|T|$ even, there always exists a T-join. A common way to construct a T-join is as follows: pairing up vertices of T and finding a path connecting them in G for each pair, and then the symmetric difference of these paths are the desired T-join. If we delete all the edges of the Eulerian cycles of a T-join, the new subgraph F becomes a forest and it remains a T-join; moreover, for every $uv \in E(G)$, $d_F(u) + d_F(v) \leqslant |T| + 2$.

In fact, T-joins associate with several well-known optimization problems: shortest paths problem instances with negative length edges, the Chinese postman problem, the 1-matching problem, and so on. In [2], Edmonds and Johnson showed that the T-join problem can be reduced to the weighted matching problem. The idea of this reduction is as follows: for every pair of vertices u, v in T, compute the distance $d(u, v)$ in G. Consider the complete graph H with vertex set T, with the edges of H weighted by the corresponding $d(u, v)$. Let M be a minimum weight perfect matching in H and, for each edge $uv \in M$, let P_{uv} be a $u - v$ path of the minimum length in G. It is not hard to show that the P_{uv}'s are mutually edge-disjoint and hence that $\cup_{uv \in M} P_{uv}$ is a minimum T-join. Edmonds [1] proved that the weighted matching problem can be solved in polynomial time. Later, Wattenhofer and Wattenhofer [8] presented an algorithm for constructing a minimum weighted perfect matching on complete graphs whose cost functions satisfy the triangular inequality, and this improved the running time to $O(n^2 \log n)$. So from algorithm complexity point of view, finding a T-join is a P-problem.

We use the notation $[x]_2 = 2\lfloor x/2 \rfloor$, i.e., $[x]_2$ denotes the maximum even number no more than x. And $[n] = \{1, 2, \ldots, n\}$. For terminology and notation not defined here, readers are referred to [7].

2 Main results

The main result presented in this paper is the following theorem.

Theorem 2.1 *Let G_1 be a connected m-fc graph and G_2 be a connected n-fc graph.*

(i) If $m \geqslant 0$, $n = 0$, $|V(G_1)| \geqslant 2m + 2$, and $|V(G_2)| \geqslant 4$, then $G_1 \boxtimes G_2$ is $(2m + 2)$-fc;

(ii) if $n = 1$, $|V(G_1)| \geqslant 2m + 3$, either $m \geqslant 3$ or $|V(G_2)| \geqslant 5$, then $G_1 \boxtimes G_2$ is $(2m + 4 - \varepsilon)$-fc, where $\varepsilon = 0$ if m is even, otherwise $\varepsilon = 1$;

(iii) if $m + 2 \leqslant |V(G_1)| \leqslant 2m + 2$, or $n + 2 \leqslant |V(G_2)| \leqslant 2n + 2$, then $G_1 \boxtimes G_2$ is mn-fc;

(iv) if $|V(G_1)| \geqslant 2m+3$ and $|V(G_2)| \geqslant 2n+3$, then $G_1 \boxtimes G_2$ is $(mn - \min\{[\frac{3m}{2}]_2, [\frac{3n}{2}]_2\})$-fc.

Remark. In [5], Györi and Imrich conjectured that the strong product of an m-extendable graph and an n-extendable graph is $([(m + 2)(n + 2)]_2 - 2)$-factor-critical. This conjecture is still open. In the above theorem, we use a stronger condition to obtain better results. For example, (iv) if G_1 and G_2 are $2m$-fc and $2n$-fc (which imply m-extendability and n-extendability), then $G_1 \boxtimes G_2$ is at least $(4mn - \min\{[3m]_2, [3n]_2\})$-fc, which is stronger than the conclusion in the conjecture when $m, n \geqslant 3$.

An important special case of the main theorem is the following, which is used many times in the proof of Theorem 2.1.

Theorem 2.2 *If G is an m-fc graph, then $G \boxtimes K_2$ is $2m$-fc.*

In addition, we need the following lemmas.

Lemma 2.3 *Let G_1 be m-fc and G_2 be n-fc ($n \geqslant 2$) such that $G_1 \boxtimes (G_2 - v)$ is $m(n-1)$-fc, for any $v \in V(G_2)$. Suppose X is a subset of $V(G_1 \boxtimes G_2)$ with $|X| = mn$. If there exists a vertex $v \in V(G_2)$ such that $|X \cap V(G_1^v)| \geqslant m$, then there is a perfect matching in $G_1 \boxtimes G_2 - X$.*

Proof. Let $X_0 = \{x_1, \ldots, x_m\}$ be any m vertices of $X \cap V(G_1^v)$. Then $G_1^v - X_0$ contains a perfect matching M as G_1 is m-fc. Consider the edges $y_1z_1, \ldots, y_pz_p$ of M such that $z_i \in X - X_0$ and $y_i \notin X - X_0$. As G_1 is m-fc, by Theorem 1.1, $\delta(G_1) \geqslant m + 1$. Thus, for $v' \in V(G_2) - v$, if $vv' \in E(G_2)$, y_i has at least $m + 2$ neighbors in $G_1^{v'}$, by the definition of strong product. Moreover, v has at least $n + 1$ neighbors in G_2 as G_2 is n-fc. Thus every vertex y_i has at least $(n+1)(m+2)$ neighbors in $G_1 \boxtimes (G_2 - v)$. Since G_1^v contains at least $m+p$ elements of X, we infer that $G_1 \boxtimes (G_2 - v) - X$ contains at least $(n+1)(m+2) - (nm - m - p) > p$ neighbors of any y_i. Thus there exist vertices $w_1, \ldots, w_p \in V(G_1 \boxtimes (G_2 - v))$ such that $w_i \notin X$, $y_iw_i \in E(G_1 \boxtimes G_2)$ for $i = 1, \ldots, p$. Let $X_1 = (X - V(G_1^v)) \cup \{w_1, \ldots, w_p\}$. Then $|X_1| \equiv m(n-1) \pmod 2$ and $|X_1| \leqslant m(n-1)$. So, there exists a perfect matching M_1 in $G_1 \boxtimes (G_2 - v) - X_1$ by the assumption. Let M_0 be the set of edges of M with both end-vertices in X. Then $M_1 \cup (M - M_0) \cup \{y_1w_1, \ldots, y_pw_p\} - \{y_1z_1, \ldots, y_pz_p\}$ is a perfect matching in $G_1 \boxtimes G_2 - X$. ∎

In the case of $n = 1$, we can deduce the following lemma by the same technique.

Lemma 2.4 *Let G_1, G_2 be m-fc and 1-fc, respectively, and let X be a subset of $V(G_1 \boxtimes G_2)$ with $|X| = 2m + 4$ when m is even (resp. $|X| = 2m + 3$ when m is odd). Suppose that v is a vertex of G_2 such that $G_1 \boxtimes (G_2 - v)$ is $(2m + 2)$-fc. Then there is a perfect matching in $G_1 \boxtimes G_2 - X$ if*

(1) *m is odd; or*

(2) *m is even and $|X \cap V(G_1^v)| \geqslant 2$.*

Although the next lemma is weaker than some of the main results, we state it for the convenience of the induction hypothesis in the proof of Theorem 2.1(iv).

Lemma 2.5 *Suppose G_1 is m-fc and G_2 is n-fc with $n \leqslant 2$. Then $G_1 \boxtimes G_2$ is mn-fc.*

Proof. It is trivial when $n = 0$. From now on, assume $n = 1$ or 2. Suppose X is a subset of $V(G_1 \boxtimes G_2)$ with $|X| = mn$.

Case 1. $|X \cap V(G_1^v)| \leqslant m$ for all $v \in V(G_2)$.

Subcase 1.1 $|X \cap V(G_1^v)| \equiv m \pmod 2$ for each $v \in V(G_2)$.

Then $G_1^v - X$ has a perfect matching as G_1 is m-fc, and hence the union of these perfect matchings is a desired perfect matching.

Subcase 1.2. $|X \cap V(G_1^v)| \equiv m+1 \pmod 2$ for some $v \in V(G_2)$.

When G_2 is 1-fc, $|V(G_2)|$ is odd. By parity, there is a vertex in G_2, say v_0, such that $|X \cap V(G_1^{v_0})| \equiv m \pmod 2$. Then $|X \cap V(G_1 \boxtimes (G_2 - v_0))|$ (resp. $|X \cap V(G_1 \boxtimes G_2)|$) is even if $n = 1$ (resp. $n = 2$). Suppose $\{v_1v_2, \ldots, v_{2t-1}v_{2t}\}$ is a perfect matching of $G_2 - v_0$ (resp. G_2) when $n = 1$ (resp. $n = 2$).

Let T denote the set of vertices v_i ($1 \leqslant i \leqslant 2t$) satisfying $|X \cap V(G_1^{v_i})| \equiv 1 \pmod 2$. Clearly, $|T|$ is even. Let F be a minimum T-join in G_2 such that $d_F(v_0)$ is as small as possible if v_0 exists. By the definition of T-join, $d_F(v_{2i-1}) + d_F(v_{2i}) + |X \cap V(G_1^{\{v_{2i-1}, v_{2i}\}})| \equiv 0 \pmod 2$ and $d_F(v_0) + |X \cap V(G_1^{v_0})| \equiv m \pmod 2$ if v_0 exists. Here we construct a matching M in $G_1 \boxtimes G_2 - X$ by considering edges of F step by step, such that one and only one edge joins $V(G_1^{v_i})$ and $V(G_1^{v_j})$ if $v_iv_j \in E(F) - \{v_1v_2, \ldots, v_{2t-1}v_{2t}\}$. If such a matching M exists, we have $|(X \cup V(M)) \cap V(G_1^{\{v_{2i-1}, v_{2i}\}})|$ ($i = 1, 2, \ldots, t$) is even and is at most $2m$, and so $G_1^{\{v_{2i-1}, v_{2i}\}} - X - V(M)$ has a perfect matching M_i by Theorem 2.2. For the vertex v_0, $G_1^{v_0} - X - V(M)$ has a perfect matching M_0 because $|(X \cup V(M)) \cap V(G_1^{v_0})| \equiv m \pmod 2$ and is less than m. Thus $\bigcup_{i=0}^{t} M_i \cup M$ (resp. $\bigcup_{i=1}^{t} M_i \cup M$) is a desired perfect matching in $G_1 \boxtimes G_2 - X$ when $n = 1$ (resp. $n = 2$).

So we only need to prove the existence of M. If for every $v_iv_j \in E(F) - \{v_1v_2, \ldots, v_{2t-1}v_{2t}\}$, there is an edge connecting $G_1^{v_i}$ and $G_1^{v_j}$ avoiding vertices in X and M constructed so far, we are done. Suppose v_iv_j is the next edge we consider. By the minimality of F, M together with X cover no more than $2m$ vertices of $G_1^{\{v_i, v_j\}}$, i.e., $|X \cap V(G_1^{\{v_i, v_j\}})| + |V(M) \cap V(G_1^{\{v_i, v_j\}})| \leqslant 2m$. Therefore, it follows from the fact that $|V(G_1)| \geqslant m + 2$, G_1 is m-connected and the definition of strong product that there is an edge between $G_1^{v_i} - X - V(M)$ and $G_1^{v_j} - X - V(M)$.

Case 2. $|X \cap V(G_1^v)| > m$ for some $v \in V(G_2)$.

In this case, $n = 2$, because Case 1 implies that $G_1 \boxtimes G_2$ is m-fc when $n = 1$. By Lemma 2.3 and above proof, there is a perfect matching in $G_1 \boxtimes G_2 - X$. ∎

3 Proofs of the Main Theorems

3.1 Proof of Theorem 2.2

Proof. Suppose $V(K_2) = \{v_1, v_2\}$, and $G \boxtimes K_2$ is not $2m$-fc. Then, by Theorem 1.2, there exists a set $S \subseteq V(G \boxtimes K_2)$ with $|S| \geqslant 2m$ such that

$$c_o(G \boxtimes K_2 - S) > |S| - 2m.$$

By parity, $c_o(G \boxtimes K_2 - S) \geqslant |S| - 2m + 2$. Note that for any vertex $u \in V(G)$, the vertices (u, v_1) and (u, v_2) have the same neighbors apart from each other and so they belong to the same component, unless we delete at least one of them. Thus, each odd component of

$G \boxtimes K_2 - S$ contains a vertex (u, v_i) $(i = 1, 2)$ with $(u, v_{3-i}) \in S$. We call (u, v_1) and (u, v_2) a *full vertex pair*.

Since there are at least $|S| - 2m + 2$ odd components, S contains at most $m - 1$ full vertex pairs, denoted by $S_1 = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_2), \ldots\}$. Then $|V(G^{v_i}) \cap S| \leqslant m - 1$ for $i = 1, 2$. Moreover, since G is m-fc and thus m-connected, $G^{v_i} - S_1$ $(i = 1, 2)$ is connected. Hence $G \boxtimes K_2 - S_1$ is connected. Let $S_2 = S - S_1$. We claim that $(G \boxtimes K_2 - S_1) - S_2$ $(= G \boxtimes K_2 - S)$ is connected, which yields a contradiction.

Claim. $(G \boxtimes K_2 - S_1) - S_2$ is connected.

Pick two vertices in $(G \boxtimes K_2 - S_1) - S_2$ arbitrarily. Suppose they are (x, v_1) and (x', v_2). It is the same when $x = x'$ or $v_1 = v_2$. Since $G \boxtimes K_2 - S_1$ is connected, there is a path connecting the two vertices, say $P = (x, v_1)(x_1, v_{i_1})(x_2, v_{i_2}) \ldots (x', v_2)$. If P contains some vertex $(x_j, v_{i_j}) \in S_2$, $i_j = 1, 2$, we know that $(x_j, v_{3-i_j}) \notin S_2$, and (x_j, v_{3-i_j}) is adjacent to vertices $(x_{j-1}, v_{i_{j-1}})$ and $(x_{j+1}, v_{i_{j+1}})$, $i_{j\pm1} = 1, 2$. So we can replace the vertex (x_j, v_{i_j}) by (x_j, v_{3-i_j}). It completes the proof. ∎

3.2 Proof of Theorem 2.1(iii)

Proof. We prove it by induction on $m + n$. When $n = 0, 1, 2$, the statement holds by Lemma 2.5. Assume it holds for smaller $m + n$. By symmetry of m and n, we assume that $m, n \geqslant 3$, and $|V(G_2)| \leqslant 2n + 2$.

Consider the strong product $G_1 \boxtimes G_2$ of an m-fc graph G_1 and an n-fc graph G_2. Let $X = \{x_1, \ldots, x_{mn}\}$ be an arbitrary set of vertices in $G_1 \boxtimes G_2$. We distinguish two cases with respect to $|X \cap V(G_1^v)|$.

Case 1. There exists a vertex v in G_2 such that $|X \cap V(G_1^v)| \geqslant m$.

By Lemma 2.3 and induction hypothesis, there is a perfect matching in $G_1 \boxtimes G_2 - X$.

Case 2. For every vertex v in G_2, $|X \cap V(G_1^v)| \leqslant m - 1$.

We only prove the subcase that both m and n are even. When m, n are odd or one of them is odd, the proofs go along the same lines.

Since G_2 is n-fc and $\delta(G_2) \geqslant n + 1 \geqslant \frac{|V(G_2)|}{2}$, by Dirac's Theorem, G_2 has a Hamilton cycle. We can pick several paths in the cycle. Every path begins with a vertex v such that $|V(G_1^v) \cap X|$ is odd and ends with another vertex v' with $|V(G_1^{v'}) \cap X|$ odd along the cycle. Let P denote the spanning subgraph of G_2 induced by the union of the edge sets of these paths. Then for every vertex v with $|V(G_1^v) \cap X|$ odd, $d_P(v)$ is 1. For every vertex v with $|V(G_1^v) \cap X|$ even, $d_P(v)$ is 0 (i.e., it is not in any path) or 2 (i.e., it is in a path). So $d_P(v) + |X \cap V(G_1^v)| \leqslant m$.

Next, construct a matching M of $|E(P)|$ edges in $G_1 \boxtimes G_2 - X$ such that one and only one edge joins $V(G_1^{v_i})$ and $V(G_1^{v_j})$ if $v_i v_j \in E(P)$ is the next edge to choose. Such an edge exists, otherwise $G_1^{\{v_i, v_j\}} - (X \cup V(M))$ is disconnected. Since M constructed so far together with X cover at most $2m - 2$ vertices of $G_1^{\{v_i, v_j\}}$, and at most $m - 1$ pair vertices like $\{(u, v_i), (u, v_j)\}$, then G_1 is disconnected after deleting at most $m - 1$ vertices, a contradiction to the fact that G_1 is m-connected by Theorem 1.1.

Then, for arbitrary $v_i \in G_2$, $G_1^{v_i} - X - V(M)$ has a perfect matching M_i, since G_1 is m-fc. So $\bigcup_{i=1}^{2t} M_i \cup M$, where $2t = |V(G_2)|$, is a perfect matching of $G_1 \boxtimes G_2 - X$. ∎

3.3 Proof of Theorem 2.1(iv)

Proof. Suppose G_1 is m-fc and G_2 is n-fc, and $|V(G_1)| \geqslant 2m+3$ and $|V(G_2)| \geqslant 2n+3$. Let X be an arbitrary subset of $V(G_1 \boxtimes G_2)$ with $|X| = mn - \min\{[\frac{3m}{2}]_2, [\frac{3n}{2}]_2\}$. If there is a vertex $v \in V(G_2)$ (or $u \in V(G_1)$) such that $|X \cap V(G_1^v)| \geqslant m$ (or $|X \cap V(G_2^u)| \geqslant n$), then it is easy to apply induction and Lemma 2.3 to complete the proof as before. So assume $|X \cap V(G_1^v)| < m$ and $|X \cap V(G_2^u)| < n$, for any $v \in V(G_2), u \in V(G_1)$.

Without loss of generality, we may assume $m \geqslant n$, and m is odd, n is even if m and n have different parities. Thus, $|X| = mn - [\frac{3n}{2}]_2$. Since G_2 is n-fc, if n is even, it has a perfect matching $M(G_2) = \{v_1v_2, \ldots, v_{2t-1}v_{2t}\}$; if n is odd, there is a vertex $v_0 \in V(G_2)$ such that $|X \cap V(G_1^{v_0})| \equiv m \pmod 2$, $|X \cap V(G_1^{v_0})| \leqslant m-2$, and $G_2 - v_0$ has a perfect matching $M(G_2 - v_0) = \{v_1v_2, \ldots, v_{2t-1}v_{2t}\}$. If $|X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| \equiv 1 \pmod 2$ for some i, put i into I_0. Clearly, $|I_0|$ is even as $|X|$ (or $|X - V(G_1^{v_0})|$) is even.

Claim. For each $i \in I_0$, put either v_{2i-1} or v_{2i} into T, and so $|T| = |I_0|$ is even. There exists a minimum T-join (T is selected over all choices of $\{v_{2i-1}, v_{2i}\}$) F of G_2 such that if $d_F(x)$ denotes the degree of x in F, then

(1) $d_F(v_{2i-1}) + d_F(v_{2i}) + |X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| \leqslant 2m$ for $i = 1, \ldots, t$;

(2) $d_F(v_{2i-1}) + d_F(v_{2i}) + |X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})|$ is even;

(3) $d_F(v_i) + d_F(v_j) + |X \cap V(G_1^{\{v_i,v_j\}})| \leqslant |V(G_1)| + m$ for $v_iv_j \in E(G_2)$;

(4) if mn is odd, $d_F(v_0) + |X \cap V(G_1^{v_0})|$ is odd and no more than m.

We show the claim by constructing F inductively. Set $I := I_0, F = \emptyset$ and $T = \{v_{2i-1}, i \in I\}$ at first. Obviously, it satisfies conditions (1), (3) and (4). Starting with $F = \emptyset$, we change F step by step so that $|I|$ decreases by two in each step. Suppose that some F has been constructed already. If $I = \emptyset$, we are done, i.e., F is the T-join required. Otherwise, select $i_0, j_0 \in I$, and set $I := I \setminus \{i_0, j_0\}$. We next show that there is a path P from v_{2i_0-1} to v_{2j_0-1}. Then, the symmetric difference of $E(P)$ and $E(F)$, maybe after deletion of some edges, is a new graph still satisfying (1), (3), (4), and at least two more indices in (2).

Obviously, the path cannot use any vertex of $\{v_{2l-1}, v_{2l}\}$ if $d_F(v_{2l-1}) + d_F(v_{2l}) + |X \cap V(G_1^{\{v_{2l-1},v_{2l}\}})| \geqslant 2m-1$ unless we join this pair to another pair when this sum is odd, and so precisely $2m-1$. But as we shall see, it is all right when we use both vertices of $\{v_{2l-1}, v_{2l}\}$ with $2m-3 \leqslant d_F(v_{2l-1}) + d_F(v_{2l}) + |X \cap V(G_1^{\{v_{2l-1},v_{2l}\}})| \leqslant 2m-2$. Similarly, the path cannot use both vertices of $\{v_i, v_j\}$ if $d_F(v_i) + d_F(v_j) + |X \cap V(G_1^{\{v_i,v_j\}})| \geqslant |V(G_1)| + m - 3$ and $v_iv_j \in E(G_2)$.

Set

$$A := \{v_{2i-1}, v_{2i} \mid 2m-1 \leqslant d_F(v_{2i-1}) + d_F(v_{2i}) + |X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| \leqslant 2m\}$$

and

$$B := \{v_i \mid d_F(v_i) + |X \cap V(G_1^{v_i})| > m + 2 - \varepsilon, v_i \in V(G_2) - A\},$$

where $\varepsilon = 1$ if mn is odd; $\varepsilon = 0$, otherwise.

Furthermore, when mn is odd, and if $d_F(v_0) + |X \cap V(G_1^{v_0})| = m$, set $A = A \cup \{v_0\}$. Let $|A| = 2a$ (resp. $|A| = 2a + 1$) if $v_0 \notin A$ (resp. $v_0 \in A$) and $|B| = b$. We consider two cases.

Case 1. $|A| + |B| \leqslant n - 1$.

Since G_2 is n-fc, it is n-connected. So $G_2 - A - B$ is connected. Thus, there is a path from $\{v_{2i_0-1}, v_{2i_0}\}$ to $\{v_{2j_0-1}, v_{2j_0}\}$ avoiding $A \cup B$. Suppose P uses both vertices of some d vertex pairs $\{v_{2l-1}, v_{2l}\}$ with $2m - 3 \leqslant d_F(v_{2l-1}) + d_F(v_{2l}) + |X \cap V(G_1^{\{v_{2l-1}, v_{2l}\}})| \leqslant 2m - 2$. Then these $2d$ vertices divide the path P into $2d + 1$ segments. Delete the edge set of the 2^{nd}, 4^{th}, $\ldots$, $2d^{th}$ segments of P. Simultaneously, if $v_{2i_0-1}v_{2i_0} \in E(P)$, replace v_{2i_0-1} in T by v_{2i_0}; if $v_{2j_0}v_{2j_0-1} \in E(P)$, replace v_{2j_0-1} in T by v_{2j_0}. The smaller edge set $E(P)$ obtained still satisfies the conditions as before and the sum of the F-degrees of the vertices v_{2l-1}, v_{2l} increases by two.

Consider the symmetric difference F_0 of the edge sets $E(P)$ and $E(F)$. If F_0 contains an Eulerian graph, then delete its edges. Trivially, F_0 defines a forest. Furthermore, F_0 remains acyclic if we add the edges $v_1v_2, \ldots, v_{2t-1}v_{2t}$ by the minimality of T-join. We only need to check (3) and (4) from now on.

For any $v_iv_j \in E(G_2)$, if $\{v_i, v_j\} \subseteq A \cup B$, then nothing changed; if $\{v_i, v_j\} \cap (A \cup B) = \emptyset$, then $d_F(v_i) + d_F(v_j) + |X \cap V(G_1^{\{v_i,v_j\}})| \leqslant 2(m + 2 - \varepsilon) \leqslant |V(G_1)| + m - 4$, and hence, $d_{F_0}(v_i) + d_{F_0}(v_j) + |X \cap V(G_1^{\{v_i,v_j\}})| \leqslant |V(G_1)| + m - 4 + 4 \leqslant |V(G_1)| + m$; otherwise, suppose $\{v_i, v_j\} \cap (A \cup B) = \{v_i\}$, then we have $d_F(v_i) + |X \cap V(G_1^{v_i})| \leqslant 2m$ by (1) and $d_F(v_j) + |X \cap V(G_1^{v_j})| \leqslant m + 2 - \varepsilon$ by the choice of B, and hence $d_{F_0}(v_i) + d_{F_0}(v_j) + |X \cap V(G_1^{\{v_i,v_j\}})| \leqslant 2m + m + 2 - \varepsilon + 2 \leqslant |V(G_1)| + m$. So (3) still holds.

Suppose mn is odd. If $v_0 \in A$, then nothing is changed; if $v_0 \notin A$, since $d_F(v_0) + |X \cap V(G_1^{v_0})| \equiv m \pmod 2$, then $d_{F_0}(v_0) + |X \cap V(G_1^{v_0})| \leqslant m - 2 + 2 = m$ and $d_{F_0}(v_0) + |X \cap V(G_1^{v_0})| \equiv m \pmod 2$. In other words, (4) holds.

Case 2. $|A| + |B| \geqslant n$.

Assume first that $v_0 \notin A$. Contracting each edge $v_{2i-1}v_{2i}$ of G_2 to a vertex w_i, G_2 is transformed into a graph H on $t + \varepsilon$ vertices, and F is transformed into F'. Note that $v_{2i-1}v_{2i} \notin E(F)$, so $d_{F'}(w_i) = d_F(v_{2i-1}) + d_F(v_{2i})$. Set $A' := \{w_i \mid \{v_{2i-1}, v_{2i}\} \subseteq A\}$, $B' := \{w_i \mid v_{2i-1} \in B \text{ or } v_{2i} \in B\}$. Then, $d_{F'}(w_i) \geqslant 2$, for $w_i \in A' \cup B'$ because of the definition of A' and B', the construction of F and the fact that $|X \cap V(G_1^{v_i})| \leqslant m - 1$.

Let z denote the number of indices i such that $|X \cap V(G_1^{\{v_{2i-1}, v_{2i}\}})|$ is odd. Then, $z \leqslant mn - [\frac{3n}{2}]_2$ and the number of leaves in F' is at most $z - 2 - \sum_{w_i \in A' \cup B'} |X \cap V(G_1^{\{v_{2i-1}, v_{2i}\}})|$ by the construction of F.

On the other hand, we have

$$\begin{aligned} & \textstyle\sum_{w_i \in A' \cup B'} (d_{F'}(w_i) - 2) \qquad (*) \\ \geqslant\ & a(2m - 3) + b(m + 1 - \varepsilon) - \textstyle\sum_{w_i \in A' \cup B'} |X \cap V(G_1^{\{v_{2i-1}, v_{2i}\}})| \\ =\ & (2a + b)m - 3a + b(1 - \varepsilon) - \textstyle\sum_{w_i \in A' \cup B'} |X \cap V(G_1^{\{v_{2i-1}, v_{2i}\}})|. \end{aligned}$$

If $2a \leqslant n$, then $(*) \geqslant mn - [\frac{3n}{2}]_2 - \sum_{w_i \in A' \cup B'} |X \cap V(G_1^{\{v_{2i-1}, v_{2i}\}})|$; otherwise, let $2a = [n+2k]_2$ ($k \geqslant 1$), then $(*) \geqslant mn - [\frac{3n}{2}]_2 + (2m-3)k - \sum_{w_i \in A' \cup B'} |X \cap V(G_1^{\{v_{2i-1}, v_{2i}\}})|$, a contradiction (note that for any forest F with at least one edge, the number of leaves is at least $\sum_{d(x) \geqslant 2, x \in F}(d(x)-2)+2$).

If mn is odd, $v_0 \in A$ and $2a+b+1 \geqslant n$, we derive the same contradiction similarly.

Contracting each edge $v_{2i-1}v_{2i}$ of G_2 to a vertex w_i, G_2 is transformed into a graph H (v_0 is unchanged and named w_0 in H) of $t+1$ vertices and F is transformed into F'. Set $A' := \{w_i \mid \{v_{2i-1}, v_{2i}\} \subseteq A\} \cup \{w_0\}$, $B' := \{w_i \mid v_{2i-1} \in B \text{ or } v_{2i} \in B\}$. Then $d_{F'}(w_i) \geqslant 2$ for $w_i \in A' \cup B'$. Similarly, F' has at most $mn - [\frac{3n}{2}]_2 - 2 - \sum_{w_i \in A' \cup B'} |X \cap V(G_1^{w_i})|$ leaves, where $G_1^{w_i}$ denotes $G_1^{\{v_{2i-1}, v_{2i}\}}$ when $i \neq 0$ or $G_1^{v_0}$ when $i = 0$.

On the other hand, we have

$$\begin{aligned} & \textstyle\sum_{w_i \in A' \cup B'}(d_{F'}(w_i)-2) \\ \geqslant\ & a(2m-3)+(m-2)+bm - \textstyle\sum_{w_i \in A' \cup B'} |X \cap V(G_1^{w_i})| \\ =\ & (2a+1+b)m-3a-2-\textstyle\sum_{w_i \in A' \cup B'} |X \cap V(G_1^{w_i})| \\ \geqslant\ & mn-[\frac{3n}{2}]_2+2-\textstyle\sum_{w_i \in A' \cup B'} |X \cap V(G_1^{w_i})|, \end{aligned}$$

a contradiction and we complete the proof of Claim.

Now, we go back to the proof of Theorem 2.1 (iv). Our aim is to construct an edge set M of $|E(F)|$ independent edges in $G_1 \boxtimes G_2 - X$ step by step. For any edge $v_iv_j \in E(F)$ (we take the edges one by one), find one and only one edge e between $V(G_1^{v_i})$ and $V(G_1^{v_j})$ such that e is not covered by X and M constructed so far, and add e into M. Suppose $v_iv_j \in E(F) \subseteq E(G_2)$ is the next edge. The vertex set $X \cap V(G_1^{\{v_i, v_j\}})$ together with the chosen edges of M cover a set Y of no more than $|V(G_1)|+m-2$ vertices by (3). If there is a vertex $u \in V(G_1)$ such that $(u, v_i), (u, v_j) \notin Y$, then add the edge $(u, v_i)(u, v_j)$ to M. Otherwise, it follows from the fact $|Y| \leqslant |V(G_1)|+m-2$ that the set Y_0 of vertices u such that both $(u, v_i), (u, v_j) \in Y$ have cardinality at most $m-2$. Set $Y_i := V(G_1^{v_i}) - Y$, $Y_j := V(G_1^{v_j}) - Y$. Condition (1) and $|V(G_1)| \geqslant 2m+3$ imply $Y_i \neq \emptyset$ and $Y_j \neq \emptyset$. Since G_1 is m-fc and hence m-connected, then there is an edge $u_1u_2 \in E(G_1)$ such that $(u_1, v_i) \in Y_i$ and $(u_2, v_j) \in Y_j$. Then add the edge $(u_1, v_i)(u_2, v_j)$ to M. Proceed similarly for other edges of F. Thus, by Claim, $|X \cap V(G_1^{\{v_{2i-1}, v_{2i}\}})| + |V(M) \cap V(G_1^{\{v_{2i-1}, v_{2i}\}})|$ is even and at most $2m$. Therefore, $G_1^{\{v_{2i-1}, v_{2i}\}} - (X \cup V(M))$ has a perfect matching M_i as $G_1 \boxtimes K_2$ is $2m$-fc, $G_1^{v_0} - (X \cup V(M))$ has a perfect matching M_0 (when mn is odd) and hence, $M \cup \bigcup_{i=1}^t M_i$ (or $M \cup \bigcup_{i=0}^t M_i$) is a desired perfect matching of $G_1 \boxtimes G_2 - X$. This completes the proof of Theorem 2.1(iv). ∎

Lemma 3.1 *Given a connected* 0*-fc graph* G*, then* $G \boxtimes K_2$ *is bicritical.*

Proof. The proof is similar to that of Theorem 2.2. ∎

Theorem 3.2 *Let* G_1, G_2 *be two connected nontrivial graphs. If* G_1 *is* 0*-fc, then* $G_1 \boxtimes G_2$ *is bicritical.*

Proof. Let $X = \{x_1, x_2\}$ be an arbitrary subset of $V(G_1 \boxtimes G_2)$. There are two cases to discuss.

Case 1. $X \subseteq V(G_1^{\{v,v'\}})$, $vv' \in E(G_2)$.

By Lemma 3.1, $G_1^{\{v,v'\}} - X$ has a perfect matching M_0. Since G_1 is 0-fc, G_1^w has a perfect matching for any $w \in V(G_2) - \{v, v'\}$, and so the union of these perfect matchings together with M_0 is a perfect matching of $G_1 \boxtimes G_2 - X$.

Case 2. $x_1 \in G_1^v, x_2 \in G_1^{v'}, vv' \notin E(G_2)$.

Suppose $x_1 = (u, v), x_2 = (u', v')$. Since G_2 is connected, there is a v-v' path in G_2, denoted by $P := v_1v_2 \dots v_k$, where $v_1 = v, v_k = v'$. Moreover, as $|X \cap V(G_1^w)| = 0$, $G_1^w - X$ has a perfect matching for any $v \in V(G_2) - \{v_1, \dots, v_k\}$. So, if we can find a perfect matching of $H = G_1 \boxtimes P - X$, we are done.

Subcase 2.1. k is even.

Set $M^* = \{(u, v_{2i})(u, v_{2i+1}) \mid 1 \leqslant i \leqslant \frac{k}{2} - 1\}$. It is easy to prove that $|(X \cup V(M^*)) \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| = 2$ for all $1 \leqslant i \leqslant \frac{k}{2}$. Then, by Lemma 3.1, $G_1^{\{v_{2i-1},v_{2i}\}} - (X \cup V(M^*))$ has a perfect matching M_i. Hence, $\bigcup_{i=1}^{\frac{k}{2}} M_i \cup M^*$ is a perfect matching of H.

Subcase 2.2. k is odd.

Suppose $M(G_1^{v_1})$ is a perfect matching of $G_1^{v_1}$ and $(u, v_1)(u'', v_1) \in M(G_1^{v_1})$. Set $M^* = \{(u'', v_1)(u'', v_2)\} \cup \{(u, v_{2i-1})(u, v_{2i}) \mid 2 \leqslant i \leqslant \frac{k-1}{2}\}$. Similarly, $|(X \cup V(M^*)) \cap V(G_1^{\{v_{2i},v_{2i+1}\}})| = 2$ for all $1 \leqslant i \leqslant \frac{k-1}{2}$, and then by Lemma 3.1, $G_1^{\{v_{2i},v_{2i+1}\}} - (X \cup V(M^*))$ has a perfect matching M_i. If $M(G_1^{v_1}) \cong M(G_1)$ denotes a perfect matching of $G_1^{v_1}$, then $\bigcup_{i=1}^{\frac{k-1}{2}} M_i \cup M(G_1^{v_1}) \cup M^* - \{(u, v_1)(u'', v_1)\}$ is the desired perfect matching of H.

This completes the proof. ∎

3.4 Proof of Theorem 2.1(i)

Proof. If both G_1 and G_2 are 0-fc (connected), it follows directly from Theorem 3.2 that $G_1 \boxtimes G_2$ is 2-fc. Next we consider when G_1 is m-fc, $m \geqslant 1$ and $|V(G_1)| \geqslant 2m + 2$.

We use induction on $|V(G_2)|$. Let X be a subset of $2m + 2$ vertices in $G_1 \boxtimes G_2$.

If $|V(G_2)| = 4$, then $P_4 \subseteq G_2$, and there is a perfect matching in $G_1 \boxtimes P_4 - X$, so is in $G_1 \boxtimes G_2 - X$. Now suppose the assertion is true for smaller $|V(G_2)|$. Since G_2 is connected and 0-fc, we may assume that G_2 has a perfect matching $M(G_2) = \{v_1v_2, \dots, v_{2t-1}v_{2t}\}$. Extend it to a spanning tree T of G_2 and contract the edges $v_1v_2, \dots, v_{2t-1}v_{2t}$ of the matching. Then T is transformed into a spanning tree of the contracted graph. Consider one of the leaves, say the vertex obtained from the contraction of v_1v_2, and v_1 has a neighbor in $\{v_3, v_4, \dots, v_{2t}\}$, say v_3. Let $X_1 \subseteq X \cap V(G_1^{\{v_1,v_2\}})$, where $|X_1| = 2m$ if $|X \cap V(G_1^{\{v_1,v_2\}})| \geqslant 2m$; $|X_1| = [|X \cap V(G_1^{\{v_1,v_2\}})|]_2$ otherwise.

Case 1. $|X_1| = 2m$.

There exists a perfect matching M_1 in $G_1^{\{v_1,v_2\}} - X_1$. Note that we can always find a matching M_1 such that the vertices in $G_1^{v_2} - X$ are matched with vertices not in $X - X_1$.

Suppose there are edges $x_1y_1, \dots, x_py_p \in M$, where $x_i \in X - X_1$, $y_i \notin X - X_1$, and $y_i \in G_1^{v_1}$. By the definition of strong product and $\delta(G_2) \geqslant m+1$, y_i has at least $m+2$ neighbors in $G_1^{v_3}$ for $i = 1, \dots, p$. Since $|X \cap V(G_1^{v_3})| \leqslant 2-p$, we can find $z_1, \dots, z_p \in V(G_1^{v_3}) - X$ such that $y_1z_1, \dots, y_pz_p \in E(G_1 \boxtimes G_2 - X)$. Let $X_2 = X \cap V(G_1 \boxtimes (G_2 - \{v_1, v_2\}))$. Then it is obvious that $|X_2 \cup \{z_1, \dots, z_p\}| \leqslant 2m+2$. Now $G_2 - \{v_1, v_2\}$ is 0-fc and connected. So, by induction hypothesis, $G_1 \boxtimes (G_2 - \{v_1, v_2\})$ is $(2m+2)$-fc, and there is a perfect matching M_2 in $G_1 \boxtimes (G_2 - \{v_1, v_2\}) - (X_2 \cup \{z_1, \dots, z_p\})$. Let M_1' denote the set of edges of M_1 whose both ends are covered by X. Then $M_1 \cup M_2 \cup \{y_1z_1, \dots, y_pz_p\} - \{x_1y_1, \dots, x_py_p\} - M_1'$ is a perfect matching in $G_1 \boxtimes G_2 - X$.

Case 2. $|X_1| < 2m$.

If $|X \cap V(G_1^{\{v_1,v_2\}})|$ is odd, we just have to choose an edge ab, where $a \in V(G_1^{v_1}) - X$ and $b \in V(G_1^{v_3}) - X$. Clearly, $G_1^{\{v_1,v_2\}} - (X_1 \cup \{a\})$ has a perfect matching M_1. Now the graph $G_2 - \{v_1, v_2\}$ is still 0-fc and connected. Let $X_2 = X \cap V(G_1 \boxtimes (G_2 - \{v_1, v_2\}))$, by induction hypothesis, there is a perfect matching M_2 in $G_1 \boxtimes (G_2 - \{v_1, v_2\}) - (X_2 \cup \{b\})$, and thus $M_1 \cup M_2 \cup \{ab\}$ is a perfect matching in $G_1 \boxtimes G_2 - X$. If $|X \cap G_1^{\{v_1,v_2\}}|$ is even, there is a perfect matching M_1 in $G_1^{\{v_1,v_2\}} - X_1$. Moreover, by induction hypothesis, there is a perfect matching M_2 in $G_1 \boxtimes (G_2 - \{v_1, v_2\}) - X$. So $M_1 \cup M_2$ is a desired perfect matching in $G_1 \boxtimes G_2$. ∎

An immediate corollary of Theorem 2.2 and Theorem 2.1(i) is the following.

Corollary 3.3 *If G_1 is m-fc with $|V(G_1)| \geqslant 2m$ and G_2 is 0-fc, then $G_1 \boxtimes G_2$ is $2m$-fc.*

3.5 Proof of Theorem 2.1(ii)

Proof. Let X be an arbitrary subset of $V(G_1 \boxtimes G_2)$ with $|X| = 2m+4-\varepsilon$, where $\varepsilon = 1$ if m is odd; $\varepsilon = 0$ otherwise. Here, we assume $m \geqslant 1$ and $|V(G_2)| \geqslant 5$ first.

Case 1. There exists a vertex, say $v_0 \in V(G_2)$, such that $|X \cap V(G_1^{v_0})| \geqslant 2-\varepsilon$.

Without loss of generality, we may assume $C_1, \dots, C_l$ are the components of $G_2 - v_0$, $l \geqslant 1$. Clearly, C_i has a perfect matching and $|X \cap V(G_1 \boxtimes C_i)| \leqslant 2m+2$ for all $1 \leqslant i \leqslant l$. If $|X \cap V(G_1 \boxtimes C_i)|$ is odd, we can join an edge between $G_1^{v_0}$ and $G_1 \boxtimes C_i$. Call such an edge set P. (It is possible that $P = \emptyset$.) Since every vertex in $G_1^{v_0}$ has at least $2(m+2)$ neighbors in each component, we can choose the endvertex of the edges of P in $G_1^{v_0}$ freely so that $G_1^{v_0} - X - V(P)$ has a perfect matching M_0. Then $|(X \cup V(P)) \cap V(G_1 \boxtimes C_i)| \leqslant 2m+2$ and is even. If $|(X \cup V(P)) \cap V(G_1 \boxtimes C_i)| \leqslant 2m$ for each i, $G_1 \boxtimes C_i - (X \cup V(P))$ has a perfect matching M_i, and therefore, $\bigcup_{i=0}^{l} M_i \cup P$ is a perfect matching in $G_1 \boxtimes G_2 - X$. Assume $|X \cap V(G_1 \boxtimes C_{i_0})| \geqslant 2m+1$. Note that $|(X \cup V(P)) \cap V(G_1 \boxtimes C_i)| \leqslant 2m$ for all $i \neq i_0$. If $|V(C_{i_0})| \geqslant 4$, by Theorem 2.1 (i), $G_1 \boxtimes C_{i_0}$ is $(2m+2)$-fc, and hence $G_1 \boxtimes C_{i_0} - X - V(P)$ has a perfect matching. If $|V(C_{i_0})| = 2$, we can reselect v_0 from C_{i_0} such that $|(X \cup V(P)) \cap V(G_1 \boxtimes C_i)| \leqslant 2m$ for every component C_i of $G_2 - v_0$. Therefore $\bigcup_{i=0}^{l} M_i \cup P$ is a perfect matching in $G_1 \boxtimes G_2 - X$.

Case 2. $|X \cap V(G_1^{v})| \leqslant 1-\varepsilon$ for all $v \in V(G_2)$.

It is easy to see that we only have to deal with the case of m even. So $m \geqslant 2$ and

$|V(G_1)| \geqslant 2m+4$.

By parity, there is at least one vertex, say $v_0 \in V(G_2)$, satisfying $|X \cap V(G_1^{v_0})| = 0$. Let $C_1, \ldots, C_l$ be the components of $G_2 - v_0$ ($l \geqslant 1$).

Subcase 2.1. $|V(G_1 \boxtimes C_i) \cap X| \leqslant 2m+2$ for $i = 1, \ldots, l$.

If $|X \cap V(G_1 \boxtimes C_i)|$ is odd for $1 \leqslant i \leqslant l$, we can join an edge between $G_1^{v_0}$ and $G_1 \boxtimes C_i$. Call such an edge set P. (It is possible that $P = \emptyset$.) Since every vertex in $G_1^{v_0}$ has at least $2(m+2)$ neighbors in each component, we can choose the endvertex of P in $G_1^{v_0}$ freely so that $G_1^{v_0} - X - V(P)$ has a perfect matching M_0. Note that $|(X \cup V(P)) \cap V(G_1 \boxtimes C_i)| \leqslant 2m+2$. Moreover, if $|V(G_1 \boxtimes C_i) \cap (X \cup V(P))| = 2m+2$, then by assumption, $|V(C_i)| \geqslant 2m+2-1 \geqslant 3$ and $|V(C_i)| \geqslant 4$ by parity. By Theorem 2.1(i) and Corollary 3.3, $G_1 \boxtimes C_i - X$ has a perfect matching M_i. Thus, $\bigcup_{i=0}^{l} M_i \cup P$ is a perfect matching in $G_1 \boxtimes G_2 - X$.

Subcase 2.2. There is a component C_1 such that $|V(G_1 \boxtimes C_1) \cap X| \geqslant 2m+3$.

There is at most one vertex of X lying in some $G_1 \boxtimes C_i$ ($i \neq 1$). Let $\{v_1v_2, \ldots, v_{2k-1}v_{2k}\}$ be a perfect matching of $G_2 - v_0$. As in the proof of Theorem 2.1(iv), we have the following Claim.

Claim. Let I_0 denote the set of indices i with $|X \cap V(G_1^{\{v_{2i-1}, v_{2i}\}})| \equiv 1 \pmod 2$. For each $i \in I_0$ put v_{2i-1} or v_{2i} into T. There exists a minimum T-join (T is selected over all choices of $\{v_{2i-1}, v_{2i}\}$) F of G_2 such that

(1) $d_F(v_0) + |X \cap V(G_1^{v_0})|$ is even and no more than m;

(2) Either there exists v_1 and v_2 such that $d_F(v_1) + d_F(v_2) + |X \cap V(G_1^{\{v_1,v_2\}})| \geqslant 2m+2$ and for $i \neq 1$, $d_F(v_{2i-1}) + d_F(v_{2i}) + |X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| \leqslant m+2 \leqslant 2m$; or $d_F(v_{2i-1}) + d_F(v_{2i}) + |X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| \leqslant 2m$ for all $1 \leqslant i \leqslant k$.

(3) For all $1 \leqslant i \leqslant k$, $d_F(v_{2i-1}) + d_F(v_{2i}) + |X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| \equiv 0 \pmod 2$.

We show the claim by constructing F inductively. Set $I := I_0, F = \emptyset$ and $T = \{v_{2i-1}, i \in I\}$ at first. Obviously, it satisfies conditions (1) and (2). Starting with $F = \emptyset$, we change F step by step so that $|I|$ decreases by two in each step. Suppose that some F has been constructed already. If $I = \emptyset$, we are done, i.e., F is the T-join required. Otherwise, select $i_0, j_0 \in I$, and set $I := I \setminus \{i_0, j_0\}$. Let P be a path from v_{2i_0-1} to v_{2j_0-1} in G_2. Moreover, if $d_F(v_0) + |X \cap V(G_1^{v_0})| = m$, P must avoid v_0; it is feasible because we can make sure that vertices $v_{2i_0-1}, v_{2i_0}, v_{2j_0-1}, v_{2j_0}$ lie in a connected component C_1 of $G_2 - v_0$. Suppose P uses both vertices of some d vertex pairs $\{v_{2i-1}, v_{2i}\}$. These $2d$ vertices divide the path into $2d+1$ segments. Delete the edge set of $2^{nd}, 4^{th}, \ldots, 2d^{th}$ segments of P. At the same time, if $v_{2i_0-1}v_{2i_0} \in E(P)$, replace v_{2i_0-1} in T by v_{2i_0}; if $v_{2j_0}v_{2j_0-1} \in E(P)$, replace v_{2j_0-1} in T by v_{2j_0}. We then obtain a smaller edge set $E(P)$. Consider the symmetric difference F_0 of $E(P)$ and $E(F)$. If F_0 contains an Eulerian graph, then delete its edges. Moreover, F_0 remains acyclic if we add the edges $v_1v_2, \ldots, v_{2k-1}v_{2k}$ by minimality of T-join.

Then the T-join F we obtained satisfies (1) and (3). We only need to check (2).

If $d_F(v_1) + d_F(v_2) + |X \cap V(G_1^{\{v_1,v_2\}})| \geqslant 2m+2$ and $d_F(v_3) + d_F(v_4) + |X \cap V(G_1^{\{v_3,v_4\}})| \geqslant m+4$, then by construction of F, easy to show that there are $(2m+2-1)+(m+4-1) > 2m+4$ in X, a contradiction. So, (2) is true, and this completes the proof of the above claim.

Now assume $d_F(v_1) + d_F(v_2) + |X \cap V(G_1^{\{v_1,v_2\}})| \geqslant 2m + 2$. It is not difficult to find a vertex set $X' \subseteq V(G_1^{\{v_1,v_2\}})$ satisfying

(i) $X \cap V(G_1^{v_i}) \subseteq X' \cap V(G_1^{v_i})$ and $|(X' - X) \cap V(G_1^{v_i})| = d_F(v_i)$, for $i = 1, 2$;

(ii) $G_1^{\{v_1,v_2\}} - X'$ has a perfect matchings.

As before, we construct a matching set M according to F. During the construction, when we take an edge with one endvertex in $G_1^{\{v_1,v_2\}}$, we choose the endvertex from $X' - X$ and pick an edge in $E(G_1 \boxtimes G_2 - X)$. It is possible because for any vertex $(u, v_i) \in X'$, it has at least $(m + 2)d_F(v_i) > m + 1$ neighbors in $G_1 \boxtimes (G_2 - \{v_1, v_2\})$.

Then $G_1^{\{v_{2i-1},v_{2i}\}} - X - V(M)$ has a perfect matching M_i for $1 \leqslant i \leqslant k$ and $G_1^{v_0}$ have a perfect matching M_0. Thus $\bigcup_{i=0}^{k} M_i \cup M$ is a perfect matching in $G_1 \boxtimes G_2 - X$.

The case that $d_F(v_{2i-1}) + d_F(v_{2i}) + |X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| \leqslant 2m$ for every $1 \leqslant i \leqslant k$ can be dealt in the same way.

Next, we consider the remaining case that $m \geqslant 3$ and $|V(G_2)| = 3$. Thus, G_2 is K_3 as G_2 is 1-fc. Let $V(G_2) = \{v_1, v_2, v_3\}$.

If there exists v_i, say v_1, such that $|X \cap V(G_1^{v_1})| \geqslant m$, then we can apply induction hypothesis on $|V(G_2)|$ as in Lemma 2.4 and thus obtain a perfect matching of $G_1 \boxtimes G_2 - X$. So, suppose $|X \cap V(G_1^{v})| < m$ for any $v \in V(G_2)$. By parity, we may assume $|X \cap V(G_1^{v_1})| \equiv m \pmod 2$, and thus, $G_1^{v_1} - X$ has a perfect matching M_1. So $|X \cap V(G_1^{\{v_2,v_3\}})| \leqslant 2m$, and $G_1^{\{v_2,v_3\}} - X$ has a perfect matching M. Hence, $G_1 \boxtimes G_2 - X$ has a perfect matching $M_1 \cup M$. It completes the proof. ∎

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