

# Uniform Star-factors of Graphs with Girth Three \*

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## Abstract

A *star-factor* of a graph  $G$  is a spanning subgraph of  $G$  such that each component of which is a star. Recently, Hartnell and Rall studied a family  $\mathcal{U}$  of graphs satisfying the property that every star-factor of a member graph has the same number of edges. They determined the family  $\mathcal{U}$  when the girth is at least five. In this paper, we investigate the family of graphs with girth three and determine all members of this family.

*Key words:* star-factor, uniform star-factor, girth, edge-weighting

## 1 Introduction

Throughout this paper, all graphs considered are simple. We refer the reader to [2] for standard graph theoretic terms not defined in this paper.

Let  $G = (V, E)$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . If  $G$  is not a forest, the length of a shortest cycle in  $G$  is called the *girth* of  $G$ . We say that a forest has an infinite girth. We shall often construct new graphs from old ones by deleting some vertices or edges. If  $W \subset V(G)$ , then  $G - W = G[V - W]$  is the subgraph of  $G$  obtained by deleting the vertices in  $W$  and all edges incident with them. Similarly, if  $E' \subset E(G)$ , then  $G - E' = (V(G), E(G) - E')$ . We denote the degree of a vertex  $x$  in  $G$  by  $d_G(x)$ , and the set of vertices adjacent to  $x$  in  $G$  by  $N_G(x)$ . A *leaf* is a vertex of degree one and a *stem* is a vertex which has at least one leaf as its neighbor. A *star* is a tree isomorphic to  $K_{1,n}$  for some  $n \geq 1$ , and the vertex of degree  $n$  is called the *center* of the star. A *star-factor* of a graph  $G$  is a spanning subgraph of  $G$  such that each component of which is a star. Clearly a graph with isolated vertices has no star-factors. It is not hard to see that every graph without isolated vertices admits a star-factor. If one limits the size of the star used, the existence of such a star-factor is non-trivial. In [1], Amahashi and Kano presented a criterion for the existence of a star-factor, i.e.,  $\{K_{1,1}, \dots, K_{1,n}\}$ -factor. Yu [4] obtained an upper bound on the number of edges in a graph with unique star-factor.

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An *edge-weighting* of a graph  $G$  is a function  $w : E(G) \rightarrow \mathbb{N}^+$ , where  $\mathbb{N}^+$  is the set of positive integers. For a subgraph  $H$ , the *weight* of  $H$  under  $w$  is the sum of all the weight values for edges belonging to  $H$ , i.e.,  $w(H) = \sum_{e \in E(H)} w(e)$ . Motivated by the minimum cost spanning tree and the optimal assignment problems, Hartnell and Rall posed an interesting general question: for a given graph, does there exist an edge-weighting function  $w$  such that a certain type of spanning subgraphs always has the same weights. In particular, they investigated the following narrow version of the problem in which the spanning subgraph is a star-factor.

**Star-Weighting Problem** (Hartnell and Rall [3]): For a given graph  $G = (V, E)$ , is there an edge-weighting  $w$  of  $G$  such that every star-factor of  $G$  has the same weights under  $w$ ?

To start the investigation, one may consider that the special case that  $w$  is a constant function, i.e., all edges in  $G$  are assigned with the same weights. In this case, every star-factor of  $G$  has the same weights if and only if all star-factors have the same number of edges. For simplicity, we assume that all edges are assigned with weight one.

We denote by  $\mathcal{U}$  the family of all graphs  $G$  such that if  $S_1$  and  $S_2$  are any two star-factors of  $G$ , then  $S_1$  and  $S_2$  have the same number of edges. Clearly,  $S_1$  and  $S_2$  have the same number of edges is equivalent to that they have the same number of components. Hartnell and Rall classified the family  $\mathcal{U}$  when graphs in  $\mathcal{U}$  have girth at least five and minimum degree at least two.

**Theorem 1.** (Hartnell and Rall [3]) *Let  $G$  be a connected graph of girth at least five and minimum degree at least two. Then all star-factors of  $G$  have the same weights if and only if  $G$  is a 5-cycle or 7-cycle.*

In this paper, we investigate the family  $\mathcal{U}$  with girth three and minimum degree at least two, and we are able to determine this family completely. The main theorem is as follows.

**Theorem 2.** *Let  $G$  be a connected graph of girth three and minimum degree at least two. Then all star-factors of  $G$  have the same weights if and only if  $G$  is one of the five graphs shown in Figure 1.*

## 2 Proof of Theorem 2

Note that if  $H$  is a spanning subgraph of  $G$ , then any star-factor of  $H$  is also a star-factor of  $G$ . The following lemma will be used frequently in reducing the problem of determining membership in  $\mathcal{U}$  to its spanning subgraphs.

**Lemma 1.** (Hartnell and Rall [3]) *Let  $F$  be a subset of  $E(G)$  such that  $G - F$  has no isolated vertices. If  $G - F$  is not in  $\mathcal{U}$ , then  $G$  is not in  $\mathcal{U}$ .*

The above lemma implies that if  $G$  is in  $\mathcal{U}$ , then so is  $G - F$ .

The idea to show that a graph does not belong to  $\mathcal{U}$  is to decompose  $G$  into several components without isolated vertices and then simply find one of them not belonging to  $\mathcal{U}$ . For the proof of Theorem 2, we shall also use the following two lemmas.

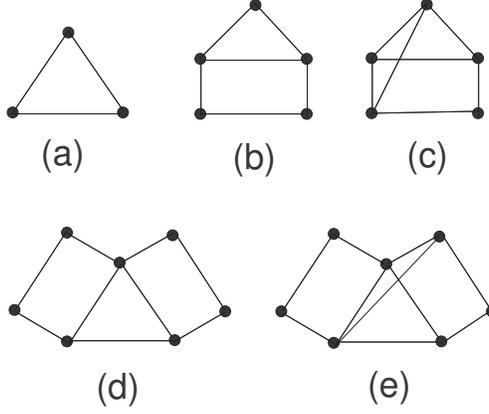


Figure 1: Graphs in  $\mathcal{U}$  with girth three and minimum degree at least two

**Lemma 2.** *Let  $G$  be a graph with a triangle such that two of its vertices are of degree two and the third is a stem, then  $G$  does not belong to  $\mathcal{U}$ .*

**Proof.** Let  $C_3 = v_1v_2v_3$  be a triangle of  $G$ , where  $v_1$  is a stem adjacent to a leaf  $u$ . Let  $S$  be a star-factor of the graph  $G - \{v_2, v_3\}$ . Note that  $v_1$  is the center of some star  $T$  in  $S$ . Let  $T'$  be the star formed from  $T$  by adding leaves  $v_2$  and  $v_3$  adjacent to  $v_1$ , and let  $S' = (S - \{T\}) \cup \{T'\}$ , then  $S'$  as well as  $S \cup \{v_2v_3\}$  are star-factors of  $G$  having different weights. Hence  $G \notin \mathcal{U}$ .  $\square$

**Lemma 3.** *Let  $G$  be a graph in  $\mathcal{U}$  with a triangle. If exactly one of the vertices on this triangle has degree at least three, then all of its neighbors that don't belong to this triangle must be stems.*

**Proof.** Suppose  $G$  is a graph satisfying the hypothesis. Let  $v$  be a vertex on the triangle of degree at least three and assume  $v$  has a neighbor  $x$  not on the triangle such that  $x$  is not a stem. By Lemma 2,  $x$  is not a leaf. Let  $F$  be the set of edges not including  $vx$  that are incident with  $x$ . The graph  $G - F$  has no isolated vertices, and the vertex  $v$  is a stem belonging to a triangle of the type that satisfies the hypothesis of Lemma 2, then  $G - F$  is not in  $\mathcal{U}$ . Thus  $G$  does not belong to  $\mathcal{U}$  by Lemma 1, a contradiction.  $\square$

Now we proceed to prove our main result.

**Proof of Theorem 2.** The only star-factor of a triangle  $C_3$  has weight two, so  $C_3 \in \mathcal{U}$ . Assume  $G$  belongs to  $\mathcal{U}$  and has girth three and minimum degree at least two but  $G$  is not a triangle. Then  $G$  contains a triangle  $C_3$  with at least two vertices of degree at least three by Lemma 3.

Let  $C_3 = v_1v_2v_3$ . We consider the following two cases.

*Case 1.*  $d_G(v_3) = 2$ ,  $d_G(v_1) \geq 3$ ,  $d_G(v_2) \geq 3$ .

Let  $F_1$  be the set of edges incident with  $v_1$  except  $v_1v_2$  and  $v_1v_3$ . Then  $G - F_1 \in \mathcal{U}$  by Lemma 1 since no isolated vertices are created in  $G - F_1$ , and all neighbors of  $v_2$  not in the triangle  $C_3$  are stems in  $G - F_1$  by Lemma 3. Let  $x$  be a neighbor of  $v_2$ , then there exists a leaf  $y$  incident with

$x$ . By the definition of  $G - F_1$ ,  $y$  is adjacent to  $v_1$  in  $G$  and  $d_G(y) = 2$ . Let  $F_2$  be the set of edges incident with  $v_2$  except  $v_2v_1$  and  $v_2v_3$ . A similar argument yields that all neighbors of  $v_1$  are stems in  $G - F_2$  and so  $y$  is a stem in  $G - F_2$ . However  $d_G(y) = 2$ , so  $x$  is the only leaf of  $y$  in  $G - F_2$ . It follows that  $d_G(x) = 2$  and  $x, y, v_1$  and  $v_2$  form a quadrangle in  $G$ . From the above discussion, we see that all neighbors of  $v_1$  and  $v_2$  except  $v_1$  and  $v_2$  are of degree two in  $G$ . Hence  $G$  is isomorphic to the graph shown in Figure 2(a) (dashed line indicates a possible edge).

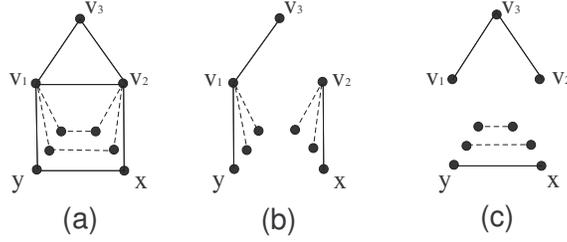


Figure 2

If  $d_G(v_1) = d_G(v_2) = 3$ , then  $G \in \mathcal{U}$ . Otherwise  $d_G(v_1) = d_G(v_2) = k$  ( $k \geq 4$ ), then  $G$  contains two star-factors shown in Figure 2(b) and 2(c), with the weights  $2k-3$  and  $k$ , respectively. However, we see  $2k-3 \neq k$  for  $k \geq 4$ , a contradiction to  $G \in \mathcal{U}$ . Hence there is exactly one quadrangle using the edge  $v_1v_2$  in  $G$ , and  $G$  is shown in Figure 1(b).

*Case 2.*  $d_G(v_i) \geq 3, i = 1, 2, 3$ .

*Claim 1.* Let  $X = \{x \in V(G) \mid N_G(x) \subseteq \{v_1, v_2, v_3\}\}$ , then  $X = \emptyset$ .

Suppose  $X \neq \emptyset$ . Let  $F_3 = \{u_1u_2 \mid u_1 \in \{v_1, v_2, v_3\}, u_2 \in V(G) - X - \{v_1, v_2, v_3\}\}$ . Then  $G - F_3$  has no isolated vertices, so  $G - F_3 \in \mathcal{U}$  by Lemma 1. Assume

$$N_G(v_1) \cap N_G(v_2) = \{x_1, \dots, x_i\},$$

$$N_G(v_2) \cap N_G(v_3) = \{y_1, \dots, y_j\},$$

$$N_G(v_1) \cap N_G(v_3) = \{z_1, \dots, z_k\},$$

$$N_G(v_1) \cap N_G(v_2) \cap N_G(v_3) = \{u_1, \dots, u_l\}.$$

Then  $G - F_3$  contains a component  $H$  with vertices in  $X$  and the triangle  $\Delta v_1v_2v_3$  shown in Figure 3(a). Since  $X \neq \emptyset$ , without loss of generality, we assume  $l = 0$  and at least one of  $i, j, k$  that is nonzero.

*Subcase 1.1.* There is exactly one of  $i, j, k$  that is nonzero. Then  $H$  can be decomposed into one or two stars.

*Subcase 1.2.* There are exactly two of  $i, j, k$  are nonzero. Assume, without loss of generality, that  $i = 0, j \neq 0$  and  $k \neq 0$ . If  $j = k = 1$ , then  $H$  can be decomposed into one or two stars. Otherwise  $H$  can be decomposed into one or two or three stars.

*Subcase 1.3.*  $i, j, k$  are all nonzero. Then  $H$  can be decomposed into two or three stars. So in all three subcases it contradicts to  $G - F_3 \in \mathcal{U}$ .

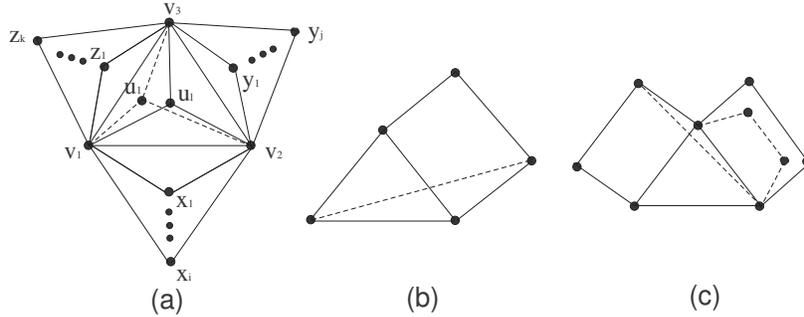


Figure 3

Let  $F_4 = \{u_1u_2 \mid u_1 \in \{v_1, v_2\}, u_2 \in V(G) - \{v_1, v_2, v_3\}\}$ . Then, by Claim 1, no isolated vertices are created in  $G - F_4$ , and all the neighbors of  $v_3$  other than  $v_1$  and  $v_2$  are stems in graph  $G - F_4$  by Lemma 3. Suppose  $u$  is a stem which is adjacent to  $v_3$  in  $G - F_4$ , and  $m$  is a leaf adjacent to  $u$  in  $G - F_4$ . It is obvious that  $m$  can only be adjacent to  $v_1$  or  $v_2$  besides  $u$  in  $G$ .

*Claim 2.*  $N_G(u) - \{m, v_3\} \subseteq \{v_1, v_2\}$  and  $N_G(u) \cap N_G(m) = \emptyset$ .

*Subcase 2.1.*  $m$  is adjacent to exactly one of  $v_1$  and  $v_2$  in  $G$ . Assume, without loss of generality, that  $m$  is adjacent to  $v_1$ , and  $u$  has other neighbors other than  $v_1, v_2, v_3$  and  $m$ . Let  $F_5 = \{u_1u_2 \mid u_1 \in \{v_2, v_3\}, u_2 \in V(G) - \{v_1, v_2, v_3\}\}$ , then  $u$  is not a leaf in  $G - F_5$ . But the neighbors of  $m$  in  $G - F_5$  are  $v_1$  and  $u$ , so  $m$  is not a stem in  $G - F_5$ . However,  $m$  should be a stem in  $G - F_5$  by Lemma 3, a contradiction. Hence  $N_G(u) - \{m, v_3\} \subseteq \{v_1, v_2\}$ . If  $u$  is also adjacent to  $v_1$ , then  $m$  and  $u$  are stems in graph  $G - F_5$  by Lemma 3, a contradiction.

*Subcase 2.2.*  $m$  is adjacent to both  $v_1$  and  $v_2$  in  $G$ . One may obtain a contradiction by a similar argument as in Subcase 2.1.

Hence  $v_1, v_2, v_3, m$  and  $u$  form at most two quadrangles with common edge  $mu$  by Claim 2, and an induced subgraph of  $G$  with the vertices  $v_1, v_2, v_3, m$  and  $u$  is isomorphic to the graph shown in Figure 3(b) (dashed line indicates a possible edge). So the subgraph  $H$  induced by all the vertices in the component, in  $G - F_4$ , which contains the 3-cycle  $C_3 = v_1v_2v_3$  is isomorphic to the graph shown in Figure 3(c).

Let  $F_6 = \{u_1u_2 \mid u_1 \in \{v_1, v_3\}, u_2 \in V(G) - \{v_1, v_2, v_3\}\}$ . By the similar argument above, both subgraphs induced by the vertices in the component of  $G - F_5$  and  $G - F_6$ , respectively, which contain the 3-cycle  $C_3 = v_1v_2v_3$  are also isomorphic to the graph shown in Figure 3(c). So  $G$  is isomorphic to the graph shown in Figure 4(a). If we delete some edges from  $G$  such that all vertices in  $G - \{v_1, v_2, v_3\}$  are of degree two, then the spanning subgraph  $G'$  of  $G$  will be the graph shown in Figure 4(b).

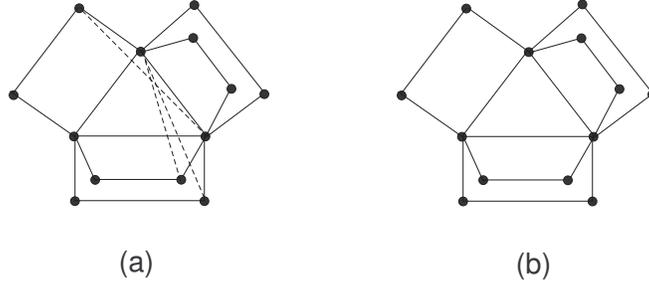


Figure 4

*Claim 3.* For each edge on the triangle  $\Delta v_1 v_2 v_3$ , there exists at most one quadrangle in  $G'$  containing it.

Let  $Y$  denote the vertices that are contained in a quadrangle which use the edge  $v_1 v_2$  and  $F_7 = \{u_1 u_2 \mid u_1 \in \{v_1, v_2, v_3\}, u_2 \in V(G) - Y - \{v_3\}\}$ . Then a component containing the triangle  $\Delta v_1 v_2 v_3$  of  $G - F_7$  is either the triangle  $\Delta v_1 v_2 v_3$  itself or a triangle satisfying the conditions in Case 1. So  $v_1 v_2$  is contained in at most one quadrangle in  $G$ . The same argument can be applied to edges  $v_2 v_3$  and  $v_1 v_3$ .

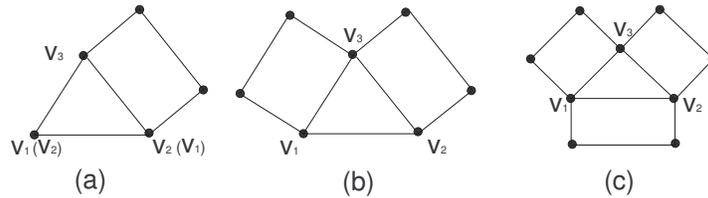


Figure 5

Now we know that  $G'$  could be the graphs shown in Figure 5. However, the graph shown in Figure 5(c) can be decomposed into three or four stars. Hence, by Lemma 1,  $G \notin \mathcal{U}$ . So the only possible graph  $G'$  are the graphs shown in Figure 5(a) and 5(b). We add the edges back following the principle of Claim 2, then  $G$  can only be the graphs shown in Figure 1(c), 1(d) and 1(e) since every vertex in the triangle  $\Delta v_1 v_2 v_3$  has degree at least three in  $G$ .

This completes the proof of Theorem 2. □

The main theorem has classified all graphs in  $\mathcal{U}$  with girth three. Combining with Theorem 1, the two families remaining to be determined are those of girth three with leaves and all those of girth four. It seems that the structures of both families are much more complicated and new techniques are required in order to determine them completely.

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