A Characterization of Graphs with Equal Domination Number and Vertex Cover Number

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Abstract

Let $\gamma(G)$ and $\beta(G)$ denote the domination number and the vertex cover number of a graph G, respectively. We use $\mathcal{G}_{\gamma=\beta}$ for the set of graphs which have equal domination number and vertex cover number. In this short note, we present a characterization for the class $\mathcal{G}_{\gamma=\beta}$.

Key words: domination number, vertex cover number, matching number

1 Introduction

In this note, we consider simple finite graphs $G = (V, E)$ only and follow [1] and [5] for terminology and definitions.

For $S \subset V(G)$, $\langle S \rangle_G$ denotes the subgraph induced by vertex set S, and $G - S$ is the subgraph of G obtained by deleting the vertices in S and all the edges incident with them. A subset S of $V(G)$ is a *dominating set* if every vertex of G is either in S or is adjacent to a vertex in S. The minimum cardinality of a dominating set is called the *domination number* and denoted by $\gamma(G)$. A set $D \subseteq V(G)$ is a *vertex cover* if every edge of G has at least one end in D. The *vertex cover number* $\beta(G)$ is the minimum cardinality of a vertex cover of G.

The class of graphs with equal domination and vertex cover number is simplify denoted by $\mathcal{G}_{\gamma=\beta}$. A characterization of the family $\mathcal{G}_{\gamma=\beta}$ with minimum degree one

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was given in [5] but was incomplete. The graph G shown in Figure 1 has domination number 4 and vertex cover number 5, respectively. However, the graph G was included in the characterization in [5]. Independently, Hartnell and Rall [2] also gave a characterization, but their characterization was involved and complicated. In this note, we give a new clear characterization of graphs in $\mathcal{G}_{\gamma=\beta}$ with minimum degree one.

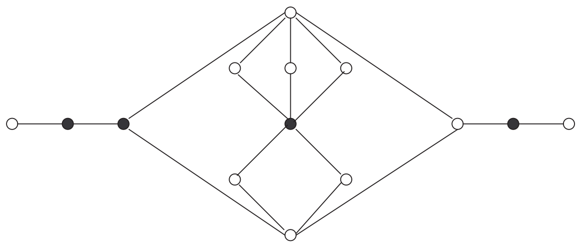

Figure 1: $\gamma(G) = 4$ and $\beta(G) = 5$

The minimum degree of G is denoted by $\delta(G)$. We denote by $I(G)$ the set of isolated vertices of G, and by $End(G)$ the set of end-vertices (i.e., vertices of degree one) of G. An edge incident with an end-vertex is called a *pendant* edge. A vertex adjacent to an end-vertex is called a *stem*, and $Stem(G)$ denotes the set of stems of G.

A graph with a single vertex is called a *trivial graph*. The *corona* $H \circ K_1$ of a graph H is the graph obtained from H by adding a pendant edge to each vertex of H. A connected graph G of order at least three is called a *generalized corona* if $V(G) = End(G) \cup Stem(G)$.

For a graph G, the maximum size of a matching is called the *matching number* of G and denoted by $\nu(G)$. The class of extremal graphs with equal domination and matching number, for abbreviation, denoted by $\mathcal{G}_{\gamma=\nu}$.

The following result is well-known.

Theorem 1. (see [3]) *If G is a graph without isolated vertices, then $\gamma(G) \leq \nu(G) \leq \beta(G)$.*

There is a characterization of the family $\mathcal{G}_{\gamma=\nu}$ in [6]. Unfortunately, their characterization is incomplete, so it was corrected in [4] as follows.

Theorem 2 (Kano, Wu and Yu [4]). *Let G be a connected graph with $\delta(G) = 1$. Then $G \in \mathcal{G}_{\gamma=\nu}$ if and only if G is K_2 or a generalized corona, or every component H of $G - (End(G) \cup Stem(G))$ is one of the following:*

(i) H is a trivial graph;
(ii) H is a connected bipartite graph with bipartition X and Y, where $1 \leq |X| < |Y|$. Let $U = V(H) \cap N_G(Stem(G))$. Then $\emptyset \neq U \subseteq Y$

and for any two distinct vertices x_1, x_2 of X that are adjacent to a common vertex of Y, there exist two distinct vertices y_1 and y_2 in $Y-U$ such that $N_H(y_i)=\{x_1,x_2\}$, for $i=1,2$;
(iii) H is isomorphic to one of graphs shown in Figure 2, and $\gamma(H-X)=\gamma(H)$ for all $\emptyset\neq X\subseteq U\subset V(H)$, where $U=V(H)\cap N_G(Stem(G))$.

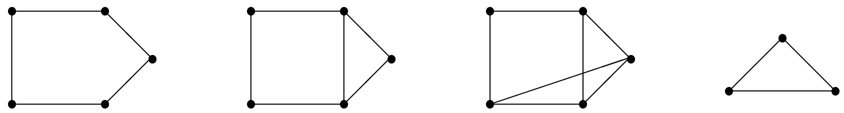

Figure 2: Graphs in (iii) of Theorem 2.

It is clear that $\mathcal{G}_{\gamma=\beta}$ is a subclass of $\mathcal{G}_{\gamma=\nu}$ from Theorem 1. Next we use Theorem 2 to give a complete characterization of graphs G with $\delta(G)=1$ in the family $\mathcal{G}_{\gamma=\beta}$.

2 Main results

We start with two lemmas, then give a clear characterization of graphs in $\mathcal{G}_{\gamma=\beta}$ with minimum degree one.

Lemma 1 (Randerath and Volkmann [5]). *Let G be a connected graph with $\delta(G)\geq 2$. Then $\gamma(G)=\beta(G)$ if and only if G is a bipartite graph with bipartition X and Y and the following property is satisfied: for any two distinct vertices x_1, x_2 of X that are adjacent to a common vertex of Y, there exist two distinct vertices y_1 and y_2 in Y such that $N_G(y_i)=\{x_1,x_2\}$ for $i=1,2$. Moreover, $\gamma(G)=\beta(G)=|X|$.*

Lemma 2 (Volkmann [7]). *Let G be a connected graph and H be a spanning subgraph of G without isolated vertices. If $\gamma(G)=\beta(G)$, then $H\in\mathcal{G}_{\gamma=\beta}$ and $\gamma(H)=\gamma(G)=\beta(G)=\beta(H)$. In particular, each component of H is in $\mathcal{G}_{\gamma=\beta}$.*

Now we give a complete characterization of graphs in $\mathcal{G}_{\gamma=\beta}$ with $\delta(G)=1$.

Theorem 3. *Let G be a connected graph with $\delta(G)=1$. Then $\gamma(G)=\beta(G)$ if and only if G is K_2 or a generalized corona, or for each component H of $G-(End(G)\cup Stem(G))$, it satisfies one of the following:*

(i) H is a trivial graph;
(ii) H is a connected bipartite graph with bipartition X and Y, where $1\leq|X|<|Y|$. Let $U_H=V(H)\cap N_G(Stem(G))$. Then $\emptyset\neq U_H\subseteq Y$ and for any two distinct vertices x_1, x_2 of X that are adjacent to a common vertex of Y, there exist two distinct vertices y_1 and y_2 in $Y-U_H$ such that $N_H(y_i)=\{x_1,x_2\}$, for $i=1,2$.

Proof. If G is K_2 or a generalized corona, then $\gamma(G) = \beta(G)$ and the theorem holds. So, in the following, we may assume that G is neither K_2 nor a generalized corona, and G has order at least three. We first show the sufficiency. Without loss of generality, assume there is a minimum vertex cover set containing all the vertices in $Stem(G)$. So

$$\beta(G) = |Stem(G)| + \sum_{H} \beta(H), \tag{1}$$

where H runs over all non-trivial components of $G - (End(G) \cup Stem(G))$.

Let $\widetilde{G}$ be a graph consisting of all the non-trivial component H of $G - (End(G) \cup Stem(G))$ and the subgraph $\langle\ End(G)\ \cup\ Stem(G)\ \cup\ I(G-(End(G)\cup Stem(G)))\ \rangle_G$. Then $\widetilde{G}$ is a spanning subgraph of G without isolated vertices. So $\gamma(H) = \beta(H) = |X|$ by Lemma 1 and Lemma 2.

Without loss of generality, for every minimum dominating set L of order $\gamma(G)$ in G, we assume $Stem(G) \subseteq L$. Let H be a non-trivial component of $G - (End(G) \cup Stem(G))$, and U_H denote the set of vertices of H dominated by $Stem(G)$, then all the vertices in $H - U_H$ are dominated by $V(H) \cap L$. By the assumption, H is a bipartite graph with bipartition X and Y, where $1 \leq |X| < |Y|$. Since $U_H \subseteq Y$, all the vertices in X of H are of degree at least two. Let $U'_H \subseteq U_H$ and $U''_H = U_H - U'_H$. Suppose $\widetilde{U} \subseteq U''_H$ is the set of vertices of degree one in graph $H - U'_H$ and $H' = \langle V(H) - U'_H \cup \widetilde{U} \rangle_H$,

Claim 1. H' is a trivial graph or all the vertices in X of graph H' are of degree at least two.

Let $x \in X$ be an isolated vertex in graph H', then x is either adjacent to at least one vertex of degree at least two in H or all neighbors of x in H are end-vertices. If it is the former case, then by assumption (ii), x has at least two neighbors in $H - U_H$, which contradicts to that x is an isolated vertex. Otherwise, $U'_H \cup \widetilde{U} = Y$ and H is a star $K_{1,n}$ $(n \geq 2)$ since H is connected. Hence if $x \in X$ is an isolated vertex in graph H' then H' is a trivial graph. Next suppose $x_1 \in X$ is a vertex of degree one in graph H' and adjacent to a vertex $y \in Y - U'_H \cup \widetilde{U}$ in H'. Since all the vertices of $Y - U'_H \cup \widetilde{U}$ in H' are of degree two, then y is adjacent to another vertex x_2 in X. By assumption (ii), there exist two distinct vertices y_1 and y_2 in $Y - U_H$ such that $N_H(y_i) = \{x_1, x_2\}$, for $i = 1, 2$. A contradiction to $d_{H'}(v) = 1$, i.e. $d_{H'}(v) \geq 2$.

Claim 2. $\gamma(H - U'_H) = \gamma(H)$, for all $U'_H \subseteq U_H$.

If H' is a trivial graph, then H is a star $K_{1,n}$ $(n \geq 2)$ by the proof of Claim 1. The claim holds. Otherwise, H' is a connected bipartite graph with minimum degree at least two and satisfies the condition of Lemma 1. So $\gamma(H') = |X|$ and X is a minimum dominating set of graph H'. Hence adding some pendant edges adjacent to vertices in X will maintain the domination number, i.e., $\gamma(H - U'_H) = |X| = \gamma(H)$.

Let $\gamma^H = \min\{\gamma(H - U'_H) \mid U'_H \subseteq U_H\}$, then $\gamma^H = \gamma(H)$ by Claim 2. Now we can compute $\gamma(G)$ as follows:

$$\begin{aligned}\gamma(G) &= |L| = |Stem(G)| + \sum_H \gamma^H \\ &= |Stem(G)| + \sum_H \gamma(H) \\ &= |Stem(G)| + \sum_H \beta(H) = \beta(G). \qquad \text{(by (1))}\end{aligned}$$

where H runs over all non-trivial components of $G - (End(G) \cup Stem(G))$.

Next we first show the necessity. Let D be a minimum vertex cover set of G with $Stem(G) \subseteq D$. Clearly, D is also a minimum dominating set of G. Let $G' = G - E(\langle Stem(G)\rangle_G)$, where $E(\langle Stem(G)\rangle_G)$ denotes the edges in the induced subgraph $\langle Stem(G)\rangle_G$. Then we next show that G' is a bipartite graph with the partite sets D and $V(G) - D$. Since G' is a spanning subgraph of G without isolated vertices, then Lemma 2 yields that $\gamma(G') = \beta(G') = |D|$. Clearly, D is also a minimum vertex cover of G' and set $V(G') - D$ is an independent set by the definition of vertex cover. Suppose that there exists an edge uv in the induced subgraph $G'[D]$. By the construction of G', there is at least one of $\{u, v\}$, say u, which is not a stem in G. But now $D - \{u\}$ is also a dominating set of G', a contradiction. Hence, G' is bipartite with bipartition D and $V(G) - D$. Consequently, each component H of $G - (End(G) \cup Stem(G))$ is a trivial graph or a bipartite graph.

Since $G \in \mathcal{G}_{\gamma=\beta}$, so G is also a member of $\mathcal{G}_{\gamma=\nu}$ by Theorem 1. From Theorem 2, we complete the proof. □

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