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Tree coloring of distance graphs with a real interval set

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Abstract

Let R be the set of real numbers and D be a subset of the positive real numbers. The *distance graph* $G(R, D)$ is a graph with the vertex set R and two vertices x and y are adjacent if and only if $|x-y| \in D$. In this work, the vertex arboricity (i.e., the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces an acyclic subgraph) of $G(R, D)$ is determined for D being an interval between 1 and δ.

Keywords: Distance graph; Vertex arboricity; Tree coloring

1. Introduction

For a graph $G = (V, E)$ and a mapping $f : V(G) \to \{1, 2, \ldots, k\}$, let $V_i = \{v \in V(G) | f(v) = i\}$. Such a mapping is often referred to as a *k-coloring* of G. Denote by $\langle V_i \rangle$ the subgraph induced by V_i in G. Depending on the graphic property enforced on each $\langle V_i \rangle$, we can define different coloring concepts. For instance, if each V_i is an independent set $(1 \le i \le k)$, then f is the well-known *proper k-coloring*. If each V_i induces a forest (i.e., each connected component of V_i is a tree), then f is called a *k-tree coloring*. Clearly, every graph has a required k-coloring if the integer k is large enough. It is interesting to find the smallest possible k such that a graph G has a required k-coloring. The minimum integer k such that G has a proper k-coloring is called the *chromatic number* of G, often denoted by $\chi(G)$. The minimum number k for which G has a k-tree coloring is called the *vertex arboricity* and denoted by $va(G)$. In other words, the vertex arboricity $va(G)$ of a graph G is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned into acyclic subgraphs. Clearly, $\chi(G) \ge va(G)$ for any graph G.

The vertex arboricity $va(G)$ has been extensively studied. For instance, Kronk and Mitchem [4] proved that $va(G) \le \lceil \frac{\Delta(G)+1}{2} \rceil$ for any graph G. Catlin and Lai [2] improved the upper bound to $va(G) \le \lceil \frac{\Delta(G)}{2} \rceil$ for a graph G being neither a cycle nor a clique. Škrekovski [5] proved that locally planar graphs have vertex arboricity ≤ 3 and that triangle-free locally planar graphs have vertex arboricity ≤ 2. Chartrand et al. [1] proved $va(K(p_1, p_2, \ldots, p_n)) = n - \max\{k | \sum_0^k p_i \le n - k\}$ for a complete n-partite graph $K(p_1, p_2, \ldots, p_n)$, where $p_0 = 0, 1 \le p_1 \le p_2 \le \cdots \le p_n$.

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Given any set D of positive real numbers, let $G(R, D)$ denote the graph whose vertices are all the points of the real number line R, such that any two vertices x, y are adjacent if and only if $|x - y| \in D$. This graph is called a *distance graph* and the set D is called the *distance set*. Coloring problems on distance graphs are motivated by the famous Hadwiger–Nelson coloring problem on the unit distance plane, which asks for the minimum number of colors necessary to color the points of the Euclidean plane (i.e., $V(G) = R^2$) such that the pairs of points with unit distance (i.e., $D = \{1\}$) are colored differently. The best known result is $4 \le \chi(G(R^2, \{1\})) \le 7$ and no substantial progress has been made on this problem for many years. Distance graphs with an interval set were introduced and studied by Eggleton et al. in 1985. In [3], it was proved that $\chi(G(R, D)) = n + 2$, where D is an interval between 1 and δ for $1 \le n < \delta \le n+1$. Recently distance graphs have been used to described various phenomena from different scientific disciplines, such as gene sequences, sequential series, on-line computing and so on.

In this note, we attempt to determine the vertex arboricity of distance graphs $G(R, D)$ with the distance set D being an interval between 1 and δ. We show that $va(G(R, D)) = n + 2$ if $1 \le n < \delta \le n + 1$.

2. Vertex arboricity of $G(R, D)$

The basic idea for determining the vertex arboricity of $G(R, D)$ is to find a subgraph of $G(R, D)$ which has a relatively simple structure but whose vertex arboricity equals $va(G(R, D))$. So, which subgraph of $G(R, D)$ is the "core structure" responsible for its vertex arboricity? The answer is a complete multipartite graph, $T(m, n)$, defined below. Since $G(R, D)$ is an infinite graph, to find a finite subgraph as a framework for this infinite graph with the same vertex arboricity is itself an interesting task.

Let $G, H_1, H_2, \ldots, H_m$ be vertex-disjoint graphs and $V(G) = \{v_1, v_2, \ldots, v_m\}$. The *composition* of G with $H_1, H_2, \ldots, H_m$, denoted by $G[H_1, H_2, \ldots, H_m]$, is the graph with the vertex set $\cup_{i=1}^{m} V(H_i)$ and the edge set consisting of $\cup_{i=1}^{m} E(H_i)$ and all edges between every vertex of H_i and every vertex of H_j if $v_i v_j \in E(G)$. The complete n-partite graph K_m^n can be expressed as $K_n[\overline{K}_m, \overline{K}_m, \ldots, \overline{K}_m]$, where K_n is the complete graph of order n and $\overline{K}_m$ is an independent set of m vertices.

Let $T(m, n) = C_{2m+1}[\overline{K}_{n+2}, K_{n+2}^n, \ldots, \overline{K}_{n+2}, K_{n+2}^n, \overline{K}_{n+2}]$, that is, $H_{2i+1} = \overline{K}_{n+2} (0 \le i \le m)$, $H_{2i} = K_{n+2}^n (1 \le i \le m)$, and have G an odd cycle C_{2m+1}. It is clear that $T(m, 1)$ is $C_{2m+1}[\overline{K}_3, \overline{K}_3, \ldots, \overline{K}_3]$ and $T(1, n)$ is a complete $(n + 2)$-partite graph K_{n+2}^{n+2}.

We need the following lemmas for our main result.

Lemma 2.1 (*Eggleton et al.* [3]). *Let D be an interval between 1 and δ and $1 \le n < \delta \le n + 1$. Then* $\chi(G(R, D)) = n + 2$.

Lemma 2.2 (*Chartrand et al.* [1]). $va(K(p_1, p_2, \ldots, p_n)) = n - \max\{k | \sum_0^k p_i \le n - k\}$ *for the complete n-partite graph $K(p_1, p_2, \ldots, p_n)$ where $p_0 = 0, 1 \le p_1 \le p_2 \le \cdots \le p_n$.*

It is clear that for each $n \ge 1$, $va(K_{n+2}^n) = n$ by Lemma 2.2.

Now we present the main result of this work.

Theorem 2.3. *Let D be an interval between 1 and δ, and $1 \le n < \delta \le n + 1$. Then $G(R, D)$ contains a subgraph $T(m, n)$ such that $va(G(R, D)) = va(T(m, n))$. Furthermore, $va(G(R, D)) = n + 2$.*

Proof. The theorem follows from the following two claims.

Claim 1. $G(R, D)$ contains a subgraph $T(m, n)$.

For $1 \le n < \delta \le n + 1$, there exists an integer m such that $n + \frac{1}{m} < \delta \le n + \frac{1}{m-1}$. We construct a subgraph $T(m, n)$ of $G(R, D)$ for $D \in \{[1, \delta], (1, \delta), (1, \delta], [1, \delta)\}$. Let $\varepsilon = \frac{\delta - (n + \frac{1}{m})}{(n+2)^2}$. Then $0 < \varepsilon \le \frac{1}{(n+2)^2 m(m-1)}$. Define vertices u_{ij}, w_{ijk} of $G(R, D)$ by

$$u_{0j} = j\frac{\varepsilon}{n+2}, \qquad \text{for } 0 \le j \le n+1,$$

$$u_{ij} = \frac{i}{m} + \varepsilon + j\frac{\varepsilon}{n+2}, \quad \text{for } 1 \le i \le m,\ 0 \le j \le n+1,$$

$$w_{ijk} = k(1 + \varepsilon) + u_{ij}, \qquad \text{for } 1 \le i \le m,\ 0 \le j \le n+1,\ 1 \le k \le n.$$

Let

$$U_i = \{u_{i0}, u_{i1}, \ldots, u_{i(n+1)}\} \quad \text{for } i = 0, 1, \ldots, m$$

and

$$W_i = \cup_{k=1}^{n} \{w_{i0k}, w_{i1k}, \ldots, w_{i(n+1)k}\} \quad \text{for } i = 1, 2, \ldots, m.$$

It is easy to see that U_i and $\{w_{i0k}, w_{i1k}, \ldots, w_{i(n+1)k}\}$ $(1 \le i \le m, 1 \le k \le n)$ are independent sets. Next, we show that the newly defined sets U_i, W_i $(i = 1, 2, \ldots, m)$ satisfy the following properties:

(1) $\langle W_i \rangle \supseteq K_{n+2}^{n}$; (2) $\langle W_i \cup U_i \rangle \supseteq K_{n+2}^{n+1}$; (3) $\langle U_{i-1} \cup W_i \rangle \supseteq K_{n+2}^{n+1}$; and finally (4) $\langle U_m \cup U_0 \rangle \supseteq K_{n+2}^{2}$.

Clearly $u_{i0} < u_{i1} < \cdots < u_{i(n+1)}$ for $0 \le i \le m$ and $w_{i0k} < w_{i1k} < \cdots < w_{i(n+1)k}$ for $1 \le i \le m, 1 \le k \le n$. The above four properties are verified below.

(1) We have $w_{i0(k+1)} - w_{i(n+1)k} = (k+1)(1+\varepsilon) + \frac{i}{m} + \varepsilon - k(1+\varepsilon) - \frac{i}{m} - \varepsilon - (n+1)\frac{\varepsilon}{n+2} = 1 + \frac{\varepsilon}{n+2} > 1$ for $k = 1, 2, \ldots, n-1$, $i = 1, 2, \ldots, m$, and $w_{i(n+1)n} - w_{i01} = n(1+\varepsilon) + \frac{i}{m} + \varepsilon + (n+1)\frac{\varepsilon}{n+2} - (1+\varepsilon) - \frac{i}{m} - \varepsilon = (n-1)(1+\varepsilon) + \frac{n+1}{n+2}\varepsilon = (n-1) + (n - \frac{1}{n+2})\varepsilon < n + \frac{1}{m} + (n+2)\varepsilon < \delta$ for $i = 1, 2, \ldots, m$. Therefore $\langle W_i \rangle \supseteq K_{n+2}^{n}$ for $i = 1, 2, \ldots, m$.

(2) In this case, $w_{i01} - u_{i(n+1)} = (1+\varepsilon) + \frac{i}{m} + \varepsilon - \frac{i}{m} - \varepsilon - (n+1)\frac{\varepsilon}{n+2} = 1 + \frac{\varepsilon}{n+2} > 1$ and $w_{i(n+1)n} - u_{i0} = n(1+\varepsilon) + \frac{i}{m} + \varepsilon + (n+1)\frac{\varepsilon}{n+2} - \frac{i}{m} - \varepsilon = n(1+\varepsilon) + \frac{n+1}{n+2}\varepsilon = n + (n + \frac{n+1}{n+2})\varepsilon < n + \frac{1}{m} + (n+2)\varepsilon < \delta$ for $i = 1, 2, \ldots, m$. Therefore $\langle W_i \cup U_i \rangle \supseteq K_{n+2}^{n+1}$ for $i = 1, 2, \ldots, m$.

(3) Similarly, we have $w_{i01} - u_{(i-1)(n+1)} = (1+\varepsilon) + \frac{i}{m} + \varepsilon - \frac{i-1}{m} - \varepsilon - (n+1)\frac{\varepsilon}{n+2} = 1 + \frac{\varepsilon}{n+2} + \frac{1}{m} > 1$ for $i = 2, \ldots, m$; $w_{101} - u_{0(n+1)} = (1+\varepsilon) + \frac{1}{m} + \varepsilon - (n+1)\frac{\varepsilon}{n+2} = 1 + \frac{n+3}{n+2}\varepsilon + \frac{1}{m} > 1$; $w_{i(n+1)n} - u_{(i-1)0} = n(1+\varepsilon) + \frac{i}{m} + \varepsilon + (n+1)\frac{\varepsilon}{n+2} - \frac{i-1}{m} - \varepsilon = n(1+\varepsilon) + \frac{1}{m} + \frac{n+1}{n+2}\varepsilon = n + \frac{1}{m} + (n + \frac{n+1}{n+2})\varepsilon < n + \frac{1}{m} + (n+2)\varepsilon < \delta$ for $i = 2, \ldots, m$ and $w_{1(n+1)n} - u_{00} = n(1+\varepsilon) + \frac{1}{m} + \varepsilon + (n+1)\frac{\varepsilon}{n+2} - 0 = n + \frac{1}{m} + (n + 1 + \frac{n+1}{n+2})\varepsilon < n + \frac{1}{m} + (n+2)\varepsilon < \delta$. Thus $\langle U_{i-1} \cup W_i \rangle \supseteq K_{n+2}^{n+1}$ for $i = 1, 2, \ldots, m$.

(4) Since $u_{m0} - u_{0(n+1)} = \frac{m}{m} + \varepsilon - (n+1)\frac{\varepsilon}{n+2} = 1 + \frac{\varepsilon}{n+2} > 1$ and $u_{m(n+1)} - u_{00} = \frac{m}{m} + \varepsilon + (n+1)\frac{\varepsilon}{n+2} - 0 = 1 + (1 + \frac{n+1}{n+2})\varepsilon < n + \frac{1}{m} + (n+2)\varepsilon < \delta$, we have $\langle U_m \cup U_0 \rangle \supseteq K_{n+2}^{2}$.

From (1)–(4), we conclude that U_i $(0 \le i \le m)$ and W_i $(1 \le i \le m)$ form the graph $T(m, n)$ in $G(R, D)$.

Claim 2. For any positive integers m and n, $va(T(m, n)) = n + 2$.

Let $U_i = V(H_{2i+1})(0 \le i \le m)$, $W_i = V(H_{2i})$ and $\langle W_i \cup U_i \rangle = G_i (1 \le i \le m)$. First, we construct an $(n+2)$-tree coloring of $T(m, n)$: let U_i be colored 0 for $0 \le i < m$ and U_m be colored $n + 1$. For $1 \le i \le m$, let n parts of W_i be colored $1, 2, \ldots, n$, respectively. It is not hard to verify that the given assignment is a tree coloring of $T(m, n)$ and so $va(T(m, n)) \le n + 2$.

We show next that $va(T(m, n)) \ge n + 2$. Otherwise, $T(m, n)$ has a $(n+1)$-tree coloring f. Let α be a color assigned the most vertices, say l_0 vertices, in U_0. Then $l_0 > 1$; otherwise there are at least $n + 2$ colors appearing in coloring f, a contradiction.

We claim that the color α would color $l_1 > 1$ vertices in U_1. Assume, to the contrary, that α colors at most one vertex in U_1; then there are at most two vertices in G_1 colored with α, so there are at least $(n+1)(n+2) - 2$ remaining vertices in G_1 that induce a complete $(n+1)$-partite graph $K(n+1, n+1, n+2, \ldots, n+2)$. By Lemma 2.2, we have

$$va(K(n+1, n+1, n+2, \ldots, n+2)) = n + 1.$$

Hence, there are at least $n + 1$ colors appearing in G_1 besides α and so there are at least $n + 2$ colors in f, a contradiction. Thus α colors $l_1 > 1$ vertices in U_1. Similarly, we conclude that α colors $l_i > 1$ vertices in U_i for $1 \le i \le m$. But these l_0 vertices in U_0 and l_m vertices in U_m induce a subgraph containing a cycle, a contradiction again.

Thus, we have $va(T(m, n)) = n + 2$.

Since $va(T(m, n)) \le va(G(R, D)) \le \chi(G(R, D)) = n + 2$ by Lemma 2.1, $T(m, n)$ is a tree chromatic subgraph of $G(R, D)$ for open interval D and consequently for half-open and closed intervals. □

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References

[1] G. Chartrand, H.V. Kronk, C.E. Wall, The point arboricity of a graph, Israel J. Math. 6 (1968) 169–175.
[2] P.A. Catlin, H.J. Lai, Vertex arboricity and maximum degree, Discrete Math. 141 (1995) 37–46.
[3] R.B. Eggleton, P. Erdös, D.K. Skilton, Colouring the real line, J. Combin. Theory, Ser. B 39 (1985) 86–100.
[4] H.V. Kronk, J. Mitchem, Critical point-arboritic graphs, J. London Math. Soc. (2) 9 (1975) 459–466.
[5] R. Škrekovski, On the critical point-arboricity graphs, J. Graph Theory 39 (2002) 50–61.