Isolated Toughness and Existence of $[a, b]$-factors in Graphs *

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Abstract

For a graph G with vertex set $V(G)$ and edge set $E(G)$, let $i(G)$ be the number of isolated vertices in G, the *isolated toughness* of G is defined as $I(G) = min\{|S|/i(G-S) \mid S \subseteq V(G), i(G-S) \geq 2\}$, if G is not complete; and $I(K_n) = n-1$. In this paper, we investigate the existence of $[a, b]$-factor in terms of this graph invariant. We proved that if G is a graph with $\delta(G) \geq a$ and $I(G) \geq a$, then G has a fractional a-factor. Moreover, if $\delta(G) \geq a$, $I(G) > (a-1) + \frac{a-1}{b}$ and $G - S$ has no $(a-1)$-regular component for any subset S of $V(G)$, then G has an $[a, b]$-factor. The later result is a generalization of Katerinis' well-known theorem about $[a, b]$-factors (P. Katerinis, Toughness of graphs and the existence of factors, *Discrete Math.* 80(1990), 81-92).

1 Introduction

All graphs considered in this paper are simple undirected graphs. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote vertex set and

*This work is sponsored by Taishan Scholar Project of Shandong, Young Scientist Award Program of Shandong, China (2006BS01015) and Natural Sciences & Engineering Research Council of Canada (Grant number OGP0122059).

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edge set of G, respectively. We use $d_G(x)$ and $\delta(G)$ to denote the degree of x in G and the minimum degree of G. For any $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. $i(G)$ and $c(G)$ are used for the number of isolated vertices and the number of components in G, respectively. For $S, T \subseteq V(G)$ let $E(S,T) = \{uv \in E(G) \mid u \in S, v \in T\}$, $E(S) = \{uv \in E(G) \mid u, v \in S\}$, and $e(S,T) = |E(S,T)|$. Other notation and terminology not defined in this paper can be found in [8].

Let $g(x) \leq f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ and let H be a spanning subgraph of G. We call H a (g, f)-*factor* of G if $g(x) \leq d_H(x) \leq f(x)$ holds for each $x \in V(G)$. Similarly, for any two nonnegative integers a and b, H is an $[a, b]$-*factor* of G if $a \leq d_H(x) \leq b$ for each $x \in V(G)$. Let $a = b = k > 0$ for each $x \in V(G)$, it is called a k-*factor*.

Fractional factors can be considered as the rationalization of the traditional factors by replacing integer-valued function by a more generous "fuzzy" function (i.e., a $[0, 1]$-valued indicator function). It defined as follows: let $h : E(G) \to [0, 1]$ be a real-value function and $d_G^h(x) = \sum_{e \in E_x} h(e)$, where $E_x = \{xy \mid xy \in E(G)\}$. Then $d_G^h(x)$ is called the *fractional degree* of x under h in G and h is called an *indictor function* if $g(x) \leq d_G^h(x) \leq f(x)$ holds for each $x \in V(G)$. Let $E^h = \{e \mid e \in E(G), h(e) \neq 0\}$ and G_h be a spanning subgraph of G such that $E(G_h) = E^h$. The G_h is referred as a *fractional* (g, f)-*factor*. The fractional k-factors and fractional $[a, b]$-factors can be defined similarly. For any function $f(x)$, we denote $f(S) = \sum_{x \in S} f(x)$.

Since fractional factors are relaxations of usual factors, many results about traditional factors also are valid for fractional factors. We quote some of them used later below.

Lemma 1.1. *(Heinrich et al. [4]) Let $g(x)$ and $f(x)$ be nonnegative integral-valued functions defined on $V(G)$. If either one of the following conditions holds*

(i) $g(x) < f(x)$ for every vertex $x \in V(G)$;

(ii) G is bipartite;

then G has a (g, f)-factor if and only if for any subset S of $V(G)$

$$g(T) - d_{G-S}(T) \leq f(S)$$

where $T = \{x \mid x \in V(G) - S,\ d_{G-S}(x) \leq g(x)\}$.

Lemma 1.2. *(Anstee [1]) Let G be a graph and let $g(x) \leq f(x)$ be two nonnegative integer-valued functions on $V(G)$. Then G has a fractional (g, f)-factor if and only if for any $S \subseteq V(G)$,*

$$g(T) - d_{G-S}(T) \leq f(S)$$

holds, where $T = \{x \mid x \in V(G) - S, d_{G-S}(x) \leq g(x)\}$.

Lemmas 1.1 and 1.2 imply that when G is bipartite or $g(x) < f(x)$ for every vertex $x \in V(G)$, then G has a fractional (g, f)-factor if and only if G has a (g, f)-factor. This is an interesting phenomenon, which indicates that a real (g, f)-factor polytope is integral under one of the given conditions.

Well-known Tutte's Theorem states that G has a 1-factor if and only if $o(G - S) \leq |S|$ for all $S \subseteq V(G)$, where $o(G - S)$ is the number of odd components in $G - S$. The following theorem provides a characterization of fractional 1-factors by using the number of isolated vertices. This result has been one of most widely quoted results in fractional factor theory.

Lemma 1.3. *(see [8]) A graph G has a fractional 1-factor if and only if $i(G - S) \leq |S|$ holds for any $S \subseteq V(G)$.*

Chvátal [2] introduced the notion of toughness for studying hamiltonian cycles and regular factors in graphs. The *toughness* of a graph G, $t(G)$, is defined as

$$t(G) = \begin{cases} min\{\frac{|S|}{c(G-S)} \mid S \subseteq V(G), c(G-S) \geq 2\} & \text{if } G \text{ is not complete;} \\ |V(G)| - 1 & \text{otherwise.} \end{cases}$$

The toughness number has become an important graph invariant for studying various fundamental properties of graphs. In particular, Chvátal conjectured that k-toughness implies a k-factor in graphs and this conjecture was confirmed positively by Enomoto et al. [3]. In this paper, we investigate the existence of fractional factors in relation to the isolated toughness of G. The parameter of isolated toughness is motivated from Chvátal's toughness by replacing $c(G - S)$ with $i(G - S)$ in the definition. The *isolated toughness*, $I(G)$, was first introduced by Ma and Liu [7] and is defined as

$$I(G) = \begin{cases} min\{\frac{|S|}{i(G-S)} \mid S \subseteq V(G), i(G-S) \geq 2\} & \text{if } G \text{ is not complete;} \\ n - 1 & \text{if } G = K_n. \end{cases}$$

Since $c(G - S) \geq i(G - S)$ for any $S \subseteq V(G)$, the isolated toughness is no less than toughness for any graph. Therefore, to weaken a condition in a theorem we could consider to replace the condition in terms of toughness by isolated toughness. In this paper, we provide several sufficient conditions in terms of isolated toughness for graphs to have fractional factors and $[a, b]$-factors. We pursue Chvátal's idea along the same line for fractional factors and prove the following

Theorem 1.1. *Let G be a graph and a be a positive integer. If $\delta(G) \geq a$ and $I(G) \geq a$, then G has a fractional a-factor.*

In [5], Katerinis obtained a sufficient condition for the existence of $[a, b]$-factor by using toughness and proved that if the toughness of G is at least $(a-1)+\frac{a}{b}$ and $a|V(G)|$ is even when $a=b$, then G has an $[a, b]$-factor. We replace the condition in Katerinis' result from toughness to isolated toughness and prove the same conclusion. Thus, we generalize this well-known result.

Theorem 1.2. *Let G be a graph with $\delta(G) \geq a$ and the isolated toughness $I(G) > (a-1)+\frac{a-1}{b}$ $(2 \leq a < b)$. If, for any subset S of $V(G)$, $G-S$ has no $(a-1)$-regular subgraph as a component, then G has an $[a, b]$-factor.*

In view of Lemmas 1.1 and 1.2, to show Theorem 1.2 we need only to prove the existence of fractional $[a, b]$-factors. So in Section 3, we only provide a proof for the following weaker version of Theorem 1.2.

Theorem 1.3. *Let G be a graph with $\delta(G) \geq a$ and the isolated toughness $I(G) > (a-1)+\frac{a-1}{b}$ $(2 \leq a < b)$. If, for any subset S of $V(G)$, $G-S$ has no $(a-1)$-regular subgraph as a component, then G has a fractional $[a, b]$-factor.*

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To do it, we require the following lemma which is a necessary and sufficient condition for existence of fractional $[a, b]$-factors given by Liu and Zhang [6].

Lemma 2.1. *Let G be a graph and $a \leq b$ be positive integers. Then G has fractional $[a, b]$-factors if and only if for any $S \subseteq V(G)$*

$$a|T| - d_{G-S}(T) \leq b|S|$$

where $T = \{x \mid x \in V(G) - S \text{ and } d_{G-S}(x) \leq a\}$.

Proof of Theorem 1.1: When $a = 1$ the result is secured by Lemma 1.3. Next we assume $a \geq 2$. If G is a complete graph with at least $a+1$ vertices, then G has a fractional a-factor by setting $h(e) = \frac{a}{|V(G)|-1}$ for each $e \in E(G)$.

Now suppose that G is not a complete and satisfies the hypothesis of the theorem, but has no fractional a-factors. By Lemma 2.1, there exists a vertex-set $S_0 \subseteq V(G)$ such that

$$a|T_0| - d_{G-S_0}(T_0) > a|S_0| \tag{2.1}$$

where $T_0 = \{x| \ x \in V(G) - S_0 \text{ and } d_{G-S_0}(x) \leq a\}$.

If $S_0 = \emptyset$, then $a|T_0| - d_G(T_0) = a|T_0| - a|T_0| = 0 > 0 = a|S_0|$ as $\delta(G) \geq a$, a contradiction. Thus $S_0 \neq \emptyset$. Let $U_0 = V(G) - (S_0 \cup T_0)$, $n = e_G(T_0, U_0)$ and $m = e_G(T_0) = e_G(T_0, T_0)$. Then (2.1) becomes

$$a|S_0| - a|T_0| + d_{G-S_0}(T_0) = a|S_0| - a|T_0| + 2m + n < 0. \tag{2.2}$$

Let A_1 be a maximal independent subset of T_0 and let $T_1 = T_0 - A_1$. Moreover, let A_i be a maximal independent subset of T_{i-1} $(2 \leq i \leq a)$, where $T_{i-1} = T_{i-2} - A_{i-1}$. Since $\triangle(G[T_0]) \leq a$, we have $A_a = \emptyset$. So we assume $2 \leq i \leq a-1$. At first we prove the following two claims.

Claim 1. $|S_0| + |T_1| + e_G(U_0, A_1) \geq a|A_1|$.

Let $U_1 = \{u \mid u \in U_0, e_G(u, A_1) > 0\}$, $U_2 = \{u \mid u \in U_0, e_G(u, A_1) = 1\}$ and $U_3 = U_1 - U_2$. Then $|U_1| \leq e_G(U_1, A_1) = e_G(U_0, A_1)$ and $|U_3| \leq |U_1|$. Let $G_1 = G - (S_0 \cup T_1 \cup U_1)$. Then $i(G_1) \geq |A_1|$ as A_1 is a maximal independent set of T_0.

Case 1. $i(G_1) \geq 2$. We have $|S_0 \cup T_1 \cup U_1| \geq a\ i(G_1) \geq a|A_1|$ by the assumption $I(G) \geq a$. On the other hand, $|S_0 \cup T_1 \cup U_1| = |S_0| + |T_1| + |U_1| \leq |S_0| + |T_1| + e_G(U_0, A_1)$. Therefore, $|S_0| + |T_1| + e_G(U_0, A_1) \geq a|A_1|$ and Claim 1 follows.

Case 2. $i(G_1) = 1$. In this case $|A_1| = 0$ or 1 as $i(G_1) \geq |A_1|$. When $|A_1| = 0$, Claim 1 clearly follows from $A_1 = T_0 = \emptyset = T_1$. When $|A_1| = 1$, let $A_1 = \{x\}$. Since $|S_0| + |T_1| + e_G(U_0, A_1) \geq d_G(x) \geq a$, we have $|S_0| + |T_1| + e_G(U_0, A_1) \geq a|A_1| = a$. Claim 1 holds.

Case 3. $i(G_1) = 0$. That is, $i(G - (S_0 \cup\ T_1 \cup U_1)) = 0$. Thus $A_1 = \emptyset$ and $T_0 = \emptyset$ as $i(G_1) \geq |A_1|$. Claim 1 holds.

Claim 2. For all i, $2 \leq i \leq a-1$, $|S_0| + e_G(T_0 - T_{i-1}, A_i) + |T_i| + e_G(U_0, A_i) \geq a|A_i|$.

If $A_i = \emptyset$, then $T_i = \emptyset$ and Claim 2 holds.

Assume that $A_i \neq \emptyset$ and let $X_i = N_G(A_i) \cap (T_0 - T_{i-1})$, $Y_i = N_G(A_i) \cup U_0$ and $G_i = G - N_G(A_i)$. Clearly, A_i is a subset of isolated vertices in G_i and $i(G_i) = i(G - N_G(A_i)) \geq |A_i|$. Consider the following cases.

Case 1. $i(G_i) \geq 2$. Since $I(G) \geq a$, we have $|S_0 \cup X_i \cup Y_i \cup T_i| \geq |N_G(A_i)| \geq a\ i(G_i) \geq a|A_i|$. On the other hand, $|S_0 \cup X_i \cup Y_i \cup T_i| \leq |S_0| + e_G(T_0 - T_{i-1}, A_i) + e_G(U_0, A_i) + |T_i|$. Therefore, $|S_0| + e_G(T_0 - T_{i-1}, A_i) + e_G(U_0, A_i) + |T_i| \geq a|A_i|$ and Claim 2 holds.

Case 2. $i(G_i) = 1$. Since $A_i \neq \emptyset$, we have $|A_i| = 1$. Let $A_i = \{x_i\}$. Then $|S_0| + e_G(T_0 - T_{i-1}, A_i) + e_G(U_0, A_i) + |T_i| \geq d_G(x_i) \geq a = a|A_i|$ and Claim 2 follows.

Combining Claim 1 and Claim 2, we have

$$\begin{array}{rl} & a|T_0| = \sum_{i=1}^{a-1} a|A_i| \\ \leq & \sum_{i=1}^{a-1} |S_0| + \sum_{i=1}^{a-1} |T_i| + \sum_{i=2}^{a-1} e_G(T_0 - T_{i-1}, A_i) + \sum_{i=1}^{a-1} e_G(U_0, A_i) \\ = & (a-1)|S_0| + \sum_{i=1}^{a-1} |T_i| + m + n. \end{array} \tag{2.3}$$

(2.2) and (2.3) imply that

$$a|S_0| + 2m + n < a|T_0| \leq (a-1)|S_0| + \sum_{i=1}^{a-1} |T_i| + m + n.$$

Thus

$$\sum_{i=1}^{a-1} |T_i| > |S_0| + m \geq m. \tag{2.4}$$

On the other hand,

$$m = e_G(T_0) = \sum_{1 \leq i < j \leq a-1} e_G(A_i, A_j) = \sum_{i=1}^{a-1} e_G(A_i, \cup_{j=i+1}^{a-1} A_j) = \sum_{j=1}^{a-1} e_G(A_j, T_j).$$

Since A_j is a maximal independent set of T_{j-1}, $e_G(A_j, x) \geq 1$ for any $x \in T_j$. Therefore $e_G(A_j, T_j) \geq |T_j|$ and $m = \sum_{j=1}^{a-1} e_G(A_j, T_j) \geq \sum_{j=1}^{a-1} |T_j|$, a contradiction to (2.4). The proof is complete. ■

3 Proof of Theorem 1.3

A *vertex covering* (or briefly, *a covering*) of a graph G is a subset C of $V(G)$ such that every edge of G has at least one end in C. To prove Theorem 1.3, we start with the following Lemma.

Lemma 3.1. *Let H be a graph with $1 \leq \delta(H) \leq \triangle(H) \leq a-1$ and $a \geq 3$ be a positive integer. Suppose that each component of H has at least one vertex of degree no more than $a-2$. Let $S_1, S_2, \cdots, S_{a-1}$ be the partition of $V(H)$, where $x \in S_j$ iff $d_H(x) = j$. Then there exists a maximal independent set I and thus a covering $C = V(H) - I$ such that*

$$\sum_{j=1}^{a-1} (a-j)c_j \leq (a-2) \sum_{j=1}^{a-1} (a-j)i_j$$

where $|I \cap S_j| = i_j$, $|C \cap S_j| = c_j$, $j = 1, 2, \cdots, a-1$.

Proof. We proceed by induction on $V(H)$.

If $|V(H)| = 2$, then $H \cong K_2$. Without loss of generality, let $H = xy$, $I = \{x\}$ and $C = V(H) - \{x\} = \{y\}$. Then

$$\sum_{j=1}^{a-1}(a-j)c_j = a-1 \leq (a-2)(a-1) = (a-2)\sum_{j=1}^{a-1}(a-j)i_j,$$

so the lemma clearly holds.

Suppose the lemma holds for $|V(H)| \leq n-1$. Now we consider $|V(H)| = n \geq 3$.

Let $m = \delta(G) = \min\{j \mid S_j \neq \emptyset\}$, so $1 \leq m \leq a-2$. Choose $y \in S_m$ and let $T = \{x \mid N_H(x) \subset N_H(y)\} \cup \{y\}$. Then $|T| \geq 1$ and T is an independent set.

If $V(H) = T \cup N_H(y)$, setting $I = T$ and $C = V(H) - T = N_H(y)$, then for any $x \in T$ we have $d_H(x) = m$ and thus

$$\sum_{j=1}^{a-1}(a-j)c_j \leq |T|m(a-m) \leq |T|(a-2)(a-m) = (a-2)\sum_{j=1}^{a-1}(a-j)i_j,$$

the lemma holds.

If $V(H) \neq T \cup N_H(y)$, let $H' = H - (T \cup N_H(y))$. Clearly, $\delta(H') \geq 1$, $\triangle(H') \leq a-1$ and for each component R of H', there is at least one vertex $v \in V(R)$ such that $d_R(v) \leq a-2$. Define $S'_j = S_j \cap V(H')$. From the induction hypothesis, there exist a maximal independent set I' and a covering set $C' = V(H') - I'$ such that

$$\sum_{j=1}^{a-1}(a-j)c'_j \leq (a-2)\sum_{j=1}^{a-1}(a-j)i'_j,$$

where $i'_j = |S'_j \cap I'|$, $c'_j = |S'_j \cap C'|$ for all $j = 1, 2, \cdots, a-1$. Let $I = I' \cup T$ and $C = V(H) - I = C' \cup N_H(y)$. Clearly, I is a maximal independent set of H. Then

$$(a-2)\sum_{j=1}^{a-1}(a-j)i_j = (a-2)\sum_{j=1}^{a-1}(a-j)i'_j + |T|(a-2)(a-m) \geq \sum_{j=1}^{a-1}(a-j)c'_j + |T|(a-2)(a-m).$$

Because $d_H(y) = m$ and $m = \min\{j \mid S_j \neq \emptyset\}$, we have

$$\sum_{j=1}^{a-1}(a-j)c_j \leq \sum_{j=1}^{a-1}(a-j)c'_j + |T|m(a-m) \leq \sum_{j=1}^{a-1}(a-j)c'_j + |T|(a-2)(a-m).$$

The induction is complete. ∎

Now we proceed to the proof of Theorem 1.3.

Proof of Theorem 1.3: If $I(G) \geq a$ and $\delta(G) \geq a$, then, by Theorem 1.1, G has a fractional a-factors. From Lemma 1.2, for any $S \subseteq V(G)$, $a|T| - d_{G-S}(T) \leq a|S| \leq b|S|$ since $a < b$, where $T = \{x \mid x \in V(G) - S \text{ and } d_{G-S}(x) \leq a-1\}$. Therefore, G has a fractional $[a,b]$-factor.

Now assume $a > I(G) \geq (a-1) + \frac{a-1}{b}$. If there exists no fractional $[a,b]$-factor in G. Then, by Lemma 1.2, there exists a vertex set $S \subset V(G)$ such that

$$a|T| - d_{G-S}(T) > b|S| \tag{3.1}$$

where $T = \{x \mid x \in V(G) - S \text{ and } d_{G-S}(x) \leq a\}$. Note that the set $T' = \{x \mid x \in V(G) - S \text{ and } d_{G-S}(x) = a\} \subseteq T$ and $a|T'| - d_{G-S}(T') = 0$, so $a|T| - d_{G-S}(T) = a|T - T'| + a|T'| - d_{G-S}(T - T') - d_{G-S}(T') = a|T - T'| - d_{G-S}(T - T')$. Therefore, in the rest of the proof, we replace T by $T - T' = \{x \mid x \in V(G) - S, d_{G-S}(x) \leq a-1\}$ instead.

If $S = \emptyset$, then clearly $T = \emptyset$ since $\delta(G) \geq a$. Thus by (3.1), we have $a|T| - d_{G-S}(T) = 0 > b|S| = 0$, a contradiction. So assume $S \neq \emptyset$ from now on.

For each i $(0 \leq i \leq a-1)$, let $T^i = \{x \mid x \in T \text{ and } d_{G-S}(x) = i\}$ and $|T^i| = t_i$. So T^0 is the set of isolated vertices. Define a subgraph $H = G[T^1 \cup T^2 \cup \cdots \cup T^{a-1}]$. By the assumption, H has no $(a-1)$-regular subgraph as a component. Since $d_H(x) = i$ for any $x \in T^i$, $\{T^i \mid i = 1, 2, \cdots a-1\}$ forms a vertex partition of H. Therefore, by Lemma 3.1, there exists a maximal independent set I and a covering $C = V(H) - I$ such that

$$\sum_{j=1}^{a-1}(a-j)c_j \leq (a-2)\sum_{j=1}^{a-1}(a-j)i_j \tag{3.2}$$

where $|I \cap T^j| = i_j$ and $|C \cap T^j| = c_j$ $(j = 1, 2, \cdots, a-1)$.

Set $W = G - (S \cup T)$, $U = S \cup C \cup (N_{G-S}(I) \cap V(W))$. Then

$$|U| \leq |S| + \sum_{j=1}^{a-1} j i_j \tag{3.3}$$

and

$$i(G-U) \geq t_0 + \sum_{j=1}^{a-1} i_j. \tag{3.4}$$

Case 1. If $i(G-U) \geq 2$, by the definition of $I(G)$, then

$$|U| \geq i(G-U)I(G). \tag{3.5}$$

Combining (3.3), (3.4) and (3.5), we obtain

$$|S| \geq I(G)(t_0 + \sum_{j=1}^{a-1} i_j) - \sum_{j=1}^{a-1} j i_j. \tag{3.6}$$

Since $a|T| - d_{G-S}(T) = at_0 + \sum_{j=1}^{a-1}(a-j)t_j$ and $t_j = i_j + c_j$. From (3.1), we have

$$at_0 + \sum_{j=1}^{a-1}(a-j)i_j + \sum_{j=1}^{a-1}(a-j)c_j > b|S|. \tag{3.7}$$

Thus (3.6) and (3.7) imply

$$\begin{aligned} & at_0 + \textstyle\sum_{j=1}^{a-1}(a-j)i_j + \sum_{j=1}^{a-1}(a-j)c_j \\ > \ & b(I(G)t_0 + I(G)\textstyle\sum_{j=1}^{a-1} i_j - \sum_{j=1}^{a-1} j i_j) \\ = \ & bI(G)t_0 + \textstyle\sum_{j=1}^{a-1}(bI(G) - bj)i_j. \end{aligned} \tag{3.8}$$

Since $I(G) > (a-1) + \frac{a-1}{b}$, we have $bI(G) - a > ab - b - 1 > 0$ as $2 \leq a < b$ and thus

$$\begin{aligned} & \textstyle\sum_{j=1}^{a-1}(a-j)c_j \\ > \ & \textstyle\sum_{j=1}^{a-1}(bI(G) - bj - a + j)i_j \\ > \ & \textstyle\sum_{j=1}^{a-1}(ab - b - 1 - bj + j)i_j. \end{aligned} \tag{3.9}$$

From (3.2) and (3.9), there exist at least one index j_0 ($1 \leq j_0 \leq a-1$) such that

$$(a-2)(a-j) > ab - b - 1 - bj + j.$$

However, this is impossible because $ab - b - 1 - bj + j - (a-2)(a-j) = (b-a-1)(a-j-1) \leq 0$ as $2 \leq a < b$ and $j \leq a-1$.

Case 2. If $i(G-U) = 0$, then by (3.4), we have $0 = i(G-U) \geq t_0 + \sum_{j=1}^{a-1} i_j$, and so $t_0 = i_j = 0$ for all $j = 1, 2, \cdots a-1$. Thus $T = \emptyset$ and $a|T| - d_{G-S}(T) = 0 > b|S| \geq b$, a contradiction.

Case 3. If $i(G-U) = 1$, by (3.4), we have $1 = i(G-U) \geq t_0 + \sum_{j=1}^{a-1} i_j$. If $t_0 = i_j = 0$ for all $j = 1, 2, \cdots, a-1$, we follow the same arguments as in the case of $i(G-U) = 0$. If $t_0 = 1$, then for all $j = 1, 2, \cdots, a-1$, $i_j = 0$ and T is an isolated vertex. Thus, by (3.1), $a|T| - d_{G-S}(T) = a > b|S| \geq b$, a contradiction. If there is some $j_0 \in \{1, 2, \cdots, a-1\}$ such that $i_{j_0} = 1$. Then H is a complete graph and $d_H(v) \leq a-2$ for each $v \in V(H)$ because $G-S$ has no $(a-1)-$regular subgraph as a component. Without loss of generality, we assume that $I = \{v\}$. Clearly, I is a maximal independent set of H. So

$$|U| = |S \cup C \cup (N_G(v) \cap V(W))| \geq |S| + d_{G-S}(v) \geq \delta(G) \geq a > I(G) \tag{3.10}$$

where $C = V(H) - I$. On the other hand, by (3.3), we have $|U| \leq |S| + j_0$. By (3.2) and (3.7), we obtain $\sum_{j=1}^{a-1}(a-j)c_j \leq (a-2)(a-j_0)$ and $b|S| < \sum_{j=1}^{a-1}(a-j)c_j + (a-j_0)$, and thus by combining these inequalities and (3.10), we have

$$\sum_{j=1}^{a-1}(a-j)c_j + (a-j_0) > b(|U| - j_0) > b(I(G) - j_0),$$

which implies that

$$(a-2)(a-j_0) > b(I(G) - j_0) - (a-j_0),$$

a contradiction due to $bI(G) - a > ba - b - 1$ and $(a - j_0 - 1)(b - a + 1) > 0$.

Hence, G has a fractional $[a, b]$-factor. ■

4 Remarks

In this section, we construct two families of graphs to show the hypotheses in Theorem 1.3 can not be weakened. So the result is the best possible.

Remark 4.1. The condition of $I(G) > (a-1) + \frac{a-1}{b}$ can not be replaced by $I(G) \geq (a-1) + \frac{a-1}{b}$. Consider a graph G constructed from K_{a-1}, bK_1 and bK_{a-1} as follows: let $V(bK_1) = \{v_1, v_2, \cdots, v_b\}$ and $\{u_1, u_2, \cdots, u_b\} \subseteq V(bK_{a-1})$ such that each u_i is from a different copy of K_{a-1}. Set $V(G) = V(K_{a-1}) \cup V(bK_1) \cup V(bK_{a-1})$ and $E(G) = \{u_iv_i \mid i = 1, 2, \cdots, b\} \cup E(bK_{a-1}) \cup E(K_{a-1}) \cup E'$, where $E' = \{uv_i \mid \forall u \in V(K_{a-1}) - \{v\}$ and $i = 2, 3, \cdots, b-1\} \cup \{vv_1\}$ for a fixed vertex v in K_{a-1}. It is not hard to see that $I(G) = (a-1) + \frac{a-1}{b}$ and for each subset $S \subseteq V(G)$, $G - S$ has no $(a-1)$-regular subgraph as a component. Choosing $S = V(K_{a-1}) - \{v\}$, then $T = \{v_2, v_3, \cdots, v_b, v\}$ if $a = 2$ and $T = \{v_1, v_2, v_3, \cdots, v_b, v\}$ if $a > 2$. In either case, we have $a|T| - d_{G-S}(T) = a(b+1) - b - 2 \geq ab - b > b(a-2) = b|S|$, this implies that G has no fractional $[a, b]$-factor.

Remark 4.2. If we delete the condition "for any subset S of $V(G)$, $G - S$ has no $(a-1)$-regular subgraph as component" in Theorem 1.3, then a graph G might not have fractional $[a, b]$-factor, even if the conditions $I(G) > (a-1) + \frac{a-1}{b}$ and $\delta(G) \geq a$ hold. Consider a graph G constructed from $K_{(n-1)(a-1)}$, $(nb+1)K_1$ and $K_{(nb+3)(a-1)}$ as follows: let $V((nb+1)K_1) = \{v_1, v_2, \cdots, v_{nb+1}\}$ and $\{u_1, u_2, \cdots, u_{nb+1}\} \subset V(K_{(nb+3)(a-1)})$. Set $V(G) = V(K_{(n-1)(a-1)}) \cup V((nb+1)K_1) \cup V(K_{(nb+3)(a-1)})$ and $E(G) = E(K_{(n-1)(a-1)}) \cup E(K_{(nb+3)(a-1)}) \cup \{u_iv_i \mid i = 1, 2, \cdots, nb+1\} \cup \{vv_i \mid v \in V(K_{(n-1)(a-1)})$ and $i = 1, 2, \cdots, nb+1\}$. It is not hard to check that

$I(G) = \frac{|V(K_{(n-1)(a-1)}) \cup V(K_{(nb+3)(a-1)})|}{|V((nb+1)K_1)|} = a-1+\frac{(n+1)(a-1)}{nb+1} > (a-1)+\frac{a-1}{b}$ and $I(G) \rightarrow (a-1)+\frac{a-1}{b}$ when $n \rightarrow \infty$. Let $S = V(K_{(n-1)(a-1)})$, then $T = V((nb+1)K_1)$ and thus $a|T| - d_{G-S}(T) = (a-1)(nb+1) > b(n-1)(a-1) = b|S|$, this implies that G has no fractional $[a,b]$-factor.

Acknowledgement: The authors are indebted to the anonymous referees for their constructive suggestions.

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