SCIENCE CHINA
Mathematics

• ARTICLES •

May 2010 Vol. 53 No. 5: 1385–1391
doi: 10.1007/s11425-009-0192-6

A conjecture on k-factor-critical and 3-γ-critical graphs

WANG Tao[1,*] & YU QingLin[2]

[1]Institute of Applied Mathematics, College of Mathematics and Information Science, Henan University, Kaifeng 475001, China;
[2]Department of Mathematics and Statistics, Thompson Rivers University, Kamloops, BC V2C5N3, Canada
Email: wangtao@henu.edu.cn, yu@tru.ca

Received March 11, 2009; accepted April 10, 2009; published online April 2, 2010

Abstract For a graph $G = (V, E)$, a subset $S \subseteq V$ is a dominating set if every vertex in V is either in S or is adjacent to a vertex in S. The domination number $\gamma(G)$ of G is the minimum order of a dominating set in G. A graph G is said to be domination vertex critical, if $\gamma(G - v) < \gamma(G)$ for any vertex v in G. A graph G is domination edge critical, if $\gamma(G \cup e) < \gamma(G)$ for any edge $e \notin E(G)$. We call a graph G k-γ-vertex-critical (resp. k-γ-edge-critical) if it is domination vertex critical (resp. domination edge critical) and $\gamma(G) = k$. Ananchuen and Plummer posed the conjecture: Let G be a k-connected graph with the minimum degree at least $k+1$, where $k \geqslant 2$ and $k \equiv |V| \pmod 2$. If G is 3-γ-edge-critical and claw-free, then G is k-factor-critical. In this paper we present a proof to this conjecture, and we also discuss the properties such as connectivity and bicriticality in 3-γ-vertex-critical claw-free graph.

Keywords domination critical graph, factor critical, bicritical

MSC(2000): 05C69

Citation: Wang T, Yu Q L. A conjecture on k-factor-critical and 3-γ-critical graphs. Sci China Math, 2010, 53(5): 1385–1391, doi: 10.1007/s11425-009-0192-6

1 Introduction

Let $G = (V, E)$ be a simple graph. A subset $S \subseteq V$ is a *dominating set* of G if every vertex in V is either in S or is adjacent to a vertex in S. If A and B are subsets of V, we say that A *dominates* B if every vertex in B has a neighbor in A or is a vertex in A; we also say that B is *dominated* by A. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set in G. A graph G is *domination vertex critical*, if $\gamma(G - v) < \gamma(G)$ for every vertex v in G. In fact, if $\gamma(G - v) < \gamma(G)$, then $\gamma(G - v) = \gamma(G) - 1$. A graph G is *domination edge critical* if $\gamma(G + e) < \gamma(G)$ for every edge $e \notin E(G)$. We say a graph G is *k-γ-vertex-critical* (resp. *k-γ-edge-critical*) if it is domination vertex critical (resp. domination edge critical) and $\gamma(G) = k$.

In this paper, we consider 3-γ-edge-critical graphs and 3-γ-vertex-critical graphs. The *neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$, and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. A graph G is called *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$.

A graph G is *k-factor-critical* if $G - S$ has a perfect matching for every k-subset S of $V(G)$. The cases of $k = 1, 2$ are referred to as *factor-critical* and *bicritical* by Gallai and Lovász [11], respectively. A *brick*

*Corresponding author

is a 3-connected bicritical graph. The factor-critical graphs are used as essential "building blocks" for the so-called Gallai-Edmonds matching structure of general graphs, and bricks are studied by Lovász to develop the brick-decomposition as a powerful tool to determine the dimension of matching lattice.

The following result was conjectured by Wojcicka [13] and was proved by Favaron et al. (cf. [8, 9, 12]).

Theorem 1.1. *Every connected 3-γ-edge-critical graph with $\delta(G) \geqslant 2$ has a hamiltonian cycle.*

According to the above theorem, if $|V|$ is odd, then G is factor-critical. For bicriticality and 3-factor-criticality, Ananchuen and Plummer proved the following two theorems.

Theorem 1.2 [1]**.** *Let G be a 2-connected 3-γ-edge-critical claw-free graph of even order. If $\delta(G) \geqslant 3$, then G is bicritical.*

Theorem 1.3 [4]**.** *Let G be a 3-connected 3-γ-edge-critical claw-free graph of odd order. If $\delta(G) \geqslant 4$, then G is 3-factor-critical.*

They posed the following conjecture in [4] based on these two theorems.

Conjecture. *Let G be a graph with $\delta(G) \geqslant k+1$ $(k \geqslant 2)$ and $k \equiv |V| \pmod 2$. If G is a k-connected 3-γ-edge-critical claw-free graph, then G is k-factor-critical.*

This conjecture is correct for $k = 2$ (Theorem 1.2) and $k = 3$ (Theorem 1.3). Their proofs are very involved and technical, they can not be extended to solve the general case of k. In this paper, we present a proof of the conjecture. In particular, when $k = 2, 3$, the proofs are much shorter than the original ones.

If $\{u, v, w\} \subset V(G)$ and $\{u, v\}$ dominates $G - w$, but $uw \notin E(G)$ and $vw \notin E(G)$, we use the notation $[u, v] \to w$.

The following lemma is obvious, but it is very handy in the proofs of results related to 3-γ-edge-critical graphs.

Lemma 1.1. *If u and v are two nonadjacent vertices in a 3-γ-edge-critical graph G, then there exists a vertex z such that either $[z, v] \to u$ or $[u, z] \to v$.*

As a conclusion of this section, we present the characterization of 2-γ-vertex-critical graphs, which will be used in the proof of the conjecture.

Theorem 1.4 [5]**.** *A graph is 2-γ-vertex-critical if and only if it is isomorphic to K_{2n} with a perfect matching removed.*

In the proof of the main theorems, we need the following characterization of k-factor-critical graphs due to Yu and Favaron, independently. Here, we denote the numbers of components and odd components in $G - S$ by $c(G - S)$ and $o(G - S)$, respectively.

Theorem 1.5 [13, 6]**.** *A graph G is k-factor-critical if and only if $o(G-S) \leqslant |S|-k$, for any $S \subseteq V(G)$ with $|S| \geqslant k$.*

2 Proof of the conjecture

Now we proceed to the proof of Ananchuen and Plummer′s conjecture.

Proof of Conjecture. Suppose, to the contrary, that G is not k-factor-critical. By Theorem 1.5, there exists a subset $S \subseteq V(G)$ with $|S| \geqslant k$ such that $o(G-S) > |S|-k$. By parity, $o(G-S) \geqslant |S|-k+2 \geqslant 2$. Choose S to be a minimal set satisfying the condition $o(G - S) \geqslant |S| - k + 2$. If $|S| \geqslant k + 1$, by the minimality of S, then it is easy to check that each vertex in S is adjacent to at least three odd components of $G - S$, which contradicts the fact that G is claw-free. Hence, $|S| = k$, S is a minimum vertex cut of G, every vertex in S is adjacent to all components of $G - S$. By claw-freedom, $G - S$ has two odd components, denoted by H_1, H_2; and $G - S$ has no even components. Let $S = \{u_1, \ldots, u_k\}$. Use $N_{H_i}(u)$ to denote the neighborhoods of u in H_i. Since S is a minimum vertex cut and G is claw-free, for any vertex u in S, $N_{H_i}(u)$ is nonnull and then is a clique. Since the minimum degree of G is at least $k + 1$, $|V(H_i)| \geqslant 3$, for $i = 1, 2$.

For the convenience of arguments, we introduce a new notion. For a vertex $x \in V(H_i)$, if there exist a vertex $v_x \in V(H_{3-i})$ and a vertex z_x such that $[v_x, z_x] \to x$, then we denote this relation in short by $v_x \to x$ and v_x is called a *reverse image* of x.

Next we define the following property:

$\mathcal{P}_i$: for any vertex $x \in V(H_i)$, there exists a vertex v_x in H_{3-i} such that $v_x \to x$.

The main idea of this proof is to show the following statement:

Assertion. *Neither property $\mathcal{P}_1$ nor property $\mathcal{P}_2$ holds in the graph G.*

By the symmetry, we need only to consider one of them, say $\mathcal{P}_2$. Suppose, to the contrary, that the property $\mathcal{P}_2$ does holds, i.e., for any vertex $x \in V(H_2)$, there exist a vertex $v_x \in V(H_1)$ and a vertex z_x such that $[v_x, z_x] \to x$.

If, for every vertex $x \in V(H_2)$, the corresponding vertex z_x lies in $V(H_2)$, then H_2 is 2-γ-vertex-critical. By Theorem 1.4, $|V(H_2)|$ is even, this contradicts the fact that H_2 is an odd component of $G - S$.

Then there exists a vertex x_1 in $V(H_2)$, such that the corresponding vertex $z_{x_1} \notin V(H_2)$. Since $V(H_2) - \{x_1\} \neq \emptyset$, the vertex z_{x_1} has to dominate $H_2 - \{x_1\}$, $z_{x_1} \in S$. Clearly, $N_{H_2}(z_{x_1}) = V(H_2) - \{x_1\}$ is a clique. Choose a vertex $x_2 \in V(H_2) \cap N_G(x_1)$. Since x_2 is adjacent to every vertex of $H_2 - \{x_2\}$, $z_{x_2} \in S$. Similarly, $N_{H_2}(z_{x_2}) = V(H_2) - \{x_2\}$ is a clique. Therefore, H_2 is complete.

Then for any vertex $x \in V(H_2)$, we have $z_x \in S$ and z_x is adjacent to each vertex in $V(H_2)$ except x, and distinct vertex x in H_2 is associated with distinct vertex z_x in S. Therefore, $k = |S| \geqslant |V(H_2)|$ (Note that if $k = 2$ we already obtain a contradiction). Let $V(H_2) = \{x_1, x_2, \ldots, x_r\}$, where r is odd. Without loss of generality, assume, for any vertex x_i, there exists a vertex $v_{x_i} \in V(H_1)$ such that $[v_{x_i}, u_i] \to x_i$.

For $i = 1, \ldots, r$, define

$$S_i = \{u \in S \mid N_G(u) \cap V(H_2) = V(H_2) - x_i\}.$$

Clearly, $S_i \neq \emptyset$, since $u_i \in S_i$. Let $S' = S - \bigcup_{i=1}^{r} S_i$. For any vertex u in S', define $\mathscr{D}_u = \{x \in V(H_2) \mid ux \notin E(G)\}$.

Claim 2.1. *If $u \in S_i$, then every vertex in $S - \{u\}$ is adjacent to either u or x_i.*

If not, let $u' \in S - \{u\}$, choose $x_i' \in N_{H_2}(u')$. Then $x_i'u', x_i'x_i, x_i'u \in E(G)$, but $\{u, x_i, u'\}$ is an independent set. So x_i' is the center of the claw induced by $\{x_i', u, x_i, u'\}$, a contradiction.

As a consequence, S_i is a clique.

Claim 2.2. *If $u \in S'$ and $|\mathscr{D}_u| \geqslant 2$, then for any vertex $x_i \in V(H_2)$ and $w \in N_{H_1}(S_i)$, then u is adjacent to either x_i or w.*

If u is not adjacent to x_i, by Claim 2.1, u dominates S_i. Because $|\mathscr{D}_u| \geqslant 2$, we can choose a vertex $x_i' \in \mathscr{D}_u - \{x_i\}$. Then x_i' dominates S_i, by claw-freedom, $uw \in E(G)$.

Claim 2.3. *For any $u_i' \in S_i$ and $u_j' \in S_j$ $(i \neq j)$, if $u_i'u_j' \in E(G)$, then $N_{H_1}(u_i') = N_{H_1}(u_j')$.*

Because $u_i'u_j' \in E(G)$, $u_i'x_j \in E(G)$, but $u_j'x_j \notin E(G)$, so for any vertex $w \in N_{H_1}(u_i')$, by claw-freedom, $wu_j' \in E(G)$, i.e., $N_{H_1}(u_i') \subseteq N_{H_1}(u_j')$. Similarly, $N_{H_1}(u_j') \subseteq N_{H_1}(u_i')$. Therefore, u_i' and u_j' have the same neighborhoods in H_1.

Claim 2.4. *It is impossible that every vertex in S is adjacent to either $x_i \in V(H_2)$ or $w \in N_{H_1}(S_i)$.*

By contradiction, assume that the statement is false. Consider the graph $G + wx_i$. From Lemma 1.1, there exists a vertex z such that either $[w, z] \to x_i$ or $[z, x_i] \to w$.

If $[w, z] \to x_i$, then $zx_i \notin E(G)$ and $z \in S$. By assumption, z is adjacent to w. Since $\{w, z\}$ dominates $G - \{x_i\}$, the vertex w is adjacent to each vertex of $H_1 - N_{H_1}(z)$. Together with the completeness of $N_{H_1}(z)$, it follows that w dominates H_1. Since every vertex in S is adjacent to either w or x_i, and x_i dominates H_2, so it implies $\{w, x_i\}$ dominates G, a contradiction.

If $[z, x_i] \to w$, then $zw \notin E(G)$. Since x_i is not adjacent to any vertex in S_i, z has to dominate S_i. Assume $w \in N_{H_1}(u_i')$ for a vertex $u_i' \in S_i$. Since $N_{H_1}(u_i')$ is a clique and $zw \notin E(G)$, $z \in S$. Since $zw \notin E(G)$, by assumption, $zx_i \in E(G)$. We know $zu_i' \in E(G)$, $zx_i \in E(G)$, by claw-freedom, for any vertex $z' \in N_{H_1}(z)$, $z'u_i' \in E(G)$. Therefore, $N_{H_1}(z) \subseteq N_{H_1}(u_i')$. The pair $\{z, x_i\}$ dominates $G - \{w\}$, so z dominates $H_1 - \{w\}$, and then u'_i dominates H_1 and H_1 is complete. By the assumption, $\{w, x_i\}$ dominates G, a contradiction.

Claim 2.5. $\bigcap_{u\in S_i} N_{H_1}(u) = \emptyset$ *for every* i, $1 \leqslant i \leqslant r$.

Assume that $\bigcap_{u\in S_i} N_{H_1}(u) \neq \emptyset$ for some i, $1 \leqslant i \leqslant r$. Let $w \in \bigcap_{u\in S_i} N_{H_1}(u)$, and let u' be any vertex in S. If $u' \in S_i$, of course $u'w \in E(G)$. If $u' \in S_j$ $(i \neq j)$, by the definition of S_j, $u'x_i \in E(G)$. If $u' \in S'$ and u' is not adjacent to x_i, then $|\mathscr{D}_{u'}| \geqslant 2$, by Claim 2.2, $u'w \in E(G)$. That is, every vertex in S is adjacent to either x_i or w, but this contradicts Claim 2.4.

From Claim 2.5, $|S_i| \geqslant 2$ for every i, $1 \leqslant i \leqslant r$. Consider $\deg_G(x_i)$, as $|S| = k$ and the minimum degree of G is at least $k + 1$, therefore,

$$k + 1 \leqslant \deg_G(x_i) \leqslant (r - 1) + (k - 2),$$

or $r \geqslant 5$ (r is odd).

Claim 2.6. *For any* $i \neq j$, *and any vertex* $u \in S_i$, *the vertex* u *does not dominate* S_j, *but it is adjacent to* S_j.

If u dominates S_j, by Claim 2.3, $\bigcap_{u'\in S_j} N_{H_1}(u') = N_{H_1}(u) \neq \emptyset$, which contradicts Claim 2.5. Then u does not dominate S_j. By a similar argument, for every vertex $v \in S_l$ $(l \neq i, j)$, v does not dominate S_j. If u is not adjacent to any vertex in S_j, then $uv \in E(G)$; otherwise, G contains a claw centered at x_t for some t, $1 \leqslant t \leqslant r$ and $t \notin \{i, j, l\}$. Thus, u is adjacent to every vertex v in S_l, by Claim 2.3,

$$\bigcap_{v\in S_l} N_{H_1}(v) = N_{H_1}(u) \neq \emptyset,$$

which contradicts Claim 2.5.

From Claim 2.6, without loss of generality, assume that $u_1u_2 \notin E(G)$. Consider $G + u_1u_2$. By Lemma 1.1, there exists a vertex z such that $[u_1, z] \to u_2$ or $[u_2, z] \to u_1$, without loss of generality, assume $[u_2, z] \to u_1$. Then $u_1z \notin E(G)$. Since $u_2x_2 \notin E(G)$, z has to dominate x_2 or $zx_2 \in E(G)$. So $z \in V(H_2) \cup S$.

If $z \in V(H_2)$, then $z = x_1$ (since $u_1z \notin E(G)$). Therefore, $\{x_1, u_2\}$ dominates $G - \{u_1\}$ and then u_2 dominates H_1. By Claim 2.1, $\{u_2, x_2\}$ dominates G, a contradiction.

Then we assume that $z \in S$. The vertex z is not adjacent to u_1, by Claim 2.1, $zx_1 \in E(G)$. Suppose z is adjacent to u_2. Since $zx_2 \in E(G)$ and $u_2x_2 \notin E(G)$, by claw-freedom, $N_{H_1}(z) \subseteq N_{H_1}(u_2)$. Since $\{z, u_2\}$ dominates $G - \{u_1\}$, so u_2 dominates H_1 and then $\{u_2, x_2\}$ dominates G, a contradiction. Therefore, z is not adjacent to u_2, and $\{u_1, u_2, z\}$ is an independent set. Both u_1 and u_2 are adjacent to every vertex in $\{x_3, \ldots, x_r\}$, by claw-freedom, z is not adjacent to any vertex in $\{x_3, \ldots, x_r\}$. Then $z \in S'$, by Claim 2.1, z dominates $S_3 \cup \cdots \cup S_r$.

From Claim 2.6, u_2 is adjacent to some vertex $u'_3 \in S_3$, and then Claim 2.3 implies $N_{H_1}(u_2) = N_{H_1}(u'_3)$. Since $u'_3z \in E(G)$, $u'_3x_4 \in E(G)$, but $zx_4 \notin E(G)$, by claw-freedom, $N_{H_1}(u'_3) \subseteq N_{H_1}(z)$, so $N_{H_1}(u_2) \subseteq N_{H_1}(z)$. As $\{z, u_2\}$ dominates $G - \{u_1\}$, z dominates H_1.

If every vertex in S is adjacent to either z or x_3, then $\{z, x_3\}$ dominates G, a contradiction. Therefore, there exists a vertex $z' \in S$ such that $N_G(z') \cap \{z, x_3\} = \emptyset$, moreover, $z' \in S'$. Thus x_3 is adjacent to at most $k - 4$ vertices in S, since x_3 is not adjacent to any vertex in $S_3 \cup \{z, z'\}$. Since the degree of x_3 is at least $k + 1$, so $k + 1 \leqslant \deg(x_3) \leqslant (k - 4) + (r - 1)$ or $r \geqslant 7$ (note that r is odd). Because x_3 dominates $S_4 \cup S_5 \cup S_6$ and z dominates $S_4 \cup S_5 \cup S_6$, by claw-freedom, z' is not adjacent to any vertex in $S_4 \cup S_5 \cup S_6$, by Claim 2.1, z' is adjacent to every vertex in $\{x_4, x_5, x_6\}$. By Claim 2.6, some vertex $u'_4 \in S_4$ and $u'_5 \in S_5$ are not adjacent, then $\{u'_4, u'_5, z'\}$ is an independent set and x_6 is the center of the claw induced by $\{x_6, u'_4, u'_5, z'\}$, a contradiction.

We complete the proof of the Assertion.

If there exists a vertex $v \in V(H_1)$ such that v is "reverse images" of all vertices in H_2, i.e., $v \to x$ for any vertex $x \in V(H_2)$, then the property $\mathcal{P}_2$ holds, a contradiction to Assertion.

Hence, for any vertex $v \in V(H_1)$, there exists a vertex $x_v \in V(H_2)$ so that $x_v \to v$. In other words, the property $\mathcal{P}_1$ holds, a contradiction again to Assertion.

Therefore we complete the proof of Conjecture. □

Remark. Since every k-factor-critical graph is $(k+1)$-edge connected and k-connected, the degree condition and connectivity condition in the theorem are necessary.

3 Connectivity in three domination critical graph

For a vertex $v \in V(G)$, we denote a minimum dominating set of $G-v$ by D_v. The following facts about D_v follow immediately from the definition of 3-γ-vertex-criticality and we shall use it frequently.

Facts. If G is a 3-γ-vertex-critical graph, then the followings hold:

(1) For any vertex v of G, $|D_v| = 2$;

(2) If $D_v = \{x, y\}$, then x and y are not adjacent to v;

(3) For every pair of distinct vertices v and w, $D_v \neq D_w$.

Lemma 3.1 [10]. *If G is a k-γ-vertex-critical graph* $(k \geqslant 2)$, *then* $\operatorname{diam}(G) \leqslant 2(k-1)$.

By the above lemma, if G is a 3-γ-vertex-critical graph, then $\operatorname{diam}(G) \leqslant 4$.

Lemma 3.2 [10]. *A connected graph G with diameter four is 3-γ-vertex critical if and only if it has two blocks, each of which is 2-γ-vertex critical.*

Lemma 3.3 [10]. *If there exist two vertices u and v in G such that $N_G[u] \subseteq N_G[v]$, then G is not k-γ-vertex-critical for any k.*

Theorem 3.1 [2]. *If G is a 3-γ-vertex-critical graph of even order, then G is 2-connected.*

We enhance the above result to the 3-connectivity by further exploring the structure of 3-γ-vertex-critical graphs.

Theorem 3.2. *Let G be a 3-γ-vertex-critical graph of even order. If the minimum degree is at least three, then G is 3-connected.*

Proof. Suppose to the contrary that G is not 3-connected. From Theorem 3.1, G has a vertex cut S with cardinality two. Let $S = \{v_1, v_2\}$ be a 2-cut of G, and $G_1, G_2, \ldots, G_m$ be the components of $G - \{v_1, v_2\}$. Since S is a inimum vertex cut, it follows that $N_{G_i}(v_j) \neq \emptyset$ for each $j = 1, 2$ and $i = 1, 2, \ldots, m$. Hence, $\delta(G) \geqslant 3$ implies that $|V(G_i)| \geqslant 2$ for each component of $G - S$. Let

$$R_i = V(G_i) - N_{G_i}(v_1) - N_{G_i}(v_2).$$

Since G is 3-γ-vertex-critical, there exists an R_i such that $R_i \neq \emptyset$; otherwise, $S = \{v_1, v_2\}$ dominates G, a contradiction. By Lemmas 3.1 and 3.2, $\operatorname{diam}(G) \leqslant 3$ and so there exists only one $R_i \neq \emptyset$. Assume that $R_1 = \emptyset$ and $R_2 \neq \emptyset$.

For any vertex $w \notin R_2$, $D_w \cap V(G_2) \neq \emptyset$ since $N_G[R_2] \subseteq V(G_2)$. Because of $V(G_1) - \{w\} \neq \emptyset$, then $D_w \cap (V(G_1) \cup \{v_1, v_2\}) \neq \emptyset$. From hereon we always assume that $D_w = \{w', z'\}$, where $w' \in V(G_1) \cup \{v_1, v_2\}$ and $z' \in V(G_2)$.

Let $X = N_{G_1}(v_1) \cap N_{G_1}(v_2)$.

Claim 3.1. *Let $w \in V(G_1)$ and $D_w = \{w', z'\}$ be defined as the above. If $w' \in V(G_1)$, then $|R_2| = 1$, say $R_2 = \{z\}$. Thus $z' = z$ and w' dominate $V(G_1) \cup \{v_1, v_2\} - \{w\}$. Moreover, $w' \in X$.*

If $w' \in V(G_1)$, then w' is not adjacent to any vertex of G_2 and so z' dominates $V(G_2)$, i.e., $V(G_2) \subseteq N_G[z']$. By Lemma 3.3, z' must be in R_2 and $|R_2| = 1$, say $R_2 = \{z\}$, so $z' = z$. Since z only dominates $V(G_2)$, so w' dominates $V(G_1) \cup \{v_1, v_2\} - \{w\}$. Clearly, $w'v_1 \in E(G)$, $w'v_2 \in E(G)$, and $w' \in X$.

Case 1. $X \neq \emptyset$.

For any vertex $x \in X$, by Fact (2), $x' \notin \{v_1, v_2\}$ and so $x' \in V(G_1)$. By Claim 3.1, $R_2 = \{z\}$, $z' = z$ dominates $V(G_2)$ and x' dominates $V(G_1) \cup \{v_1, v_2\} - \{x\}$, moreover, $x' \in X$. Therefore, $G[X]$ is 2-γ-vertex-critical. Let $G[X]$ be $K_{|X|}$ with a perfect matching $F = \{x_1x_2, x_3x_4, \ldots, x_{2r-1}x_{2r}\}$ removed (see Theorem 1.4). Then every vertex $x \in X$ dominates $V(G_1) - \{x'\}$. Since $\{x_2, z\}$ dominate $G - x_1$, then $G - S$ has only two components G_1 and G_2.

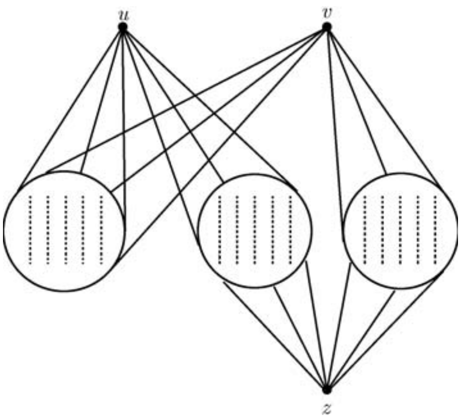

Figure 1 $X = V(G_1)$

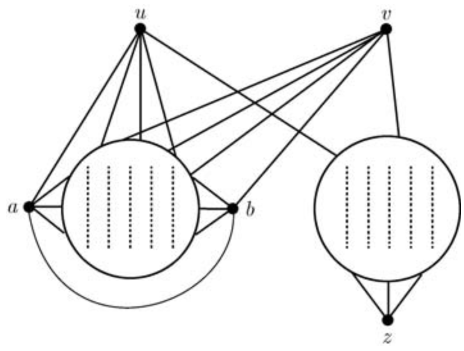

Figure 2 $X \neq V(G_1)$

Case 1.1. $V(G_1) = X$.

If there exists a vertex $w \in N_{G_2}(v_1) \cap N_{G_2}(v_2)$, by Fact (2), $D_w \cap \{v_1, v_2\} = \emptyset$. Then $w' \in V(G_1)$ and w' dominates $V(G_1)$, contradicting to $\gamma(G_1) = 2$, therefore, $N_{G_2}(v_1) \cap N_{G_2}(v_2) = \emptyset$. For any vertex $w \in N_{G_2}(v_1)$, by Fact (2), $D_w \cap \{v_1, z\} = \emptyset$. Since w' has to dominate $V(G_1)$, and G_1 is 2-γ-vertex-critical, so $w' = v_2$. By Lemma 3.3, $v_1v_2 \notin E(G)$. Then $D_w \cap V(G_2)$ has to dominate v_1, i.e., $D_w \cap V(G_2) \in N_{G_2}(v_1)$. Therefore, $N_{G_2}(v_1)$ is 2-γ-vertex-critical. Similarly, $N_{G_2}(v_2)$ is 2-γ-vertex-critical. By Theorem 1.4, $|V(G)|$ is odd, a contradiction (see Figure 1).

Case 1.2. $V(G_1) \neq X$.

Assume that $N_{G_1}(v_1) - X \neq \emptyset$. Let $a \in N_{G_1}(v_1) - X$. Consider D_a, by Fact (2), $D_a \cap (X \cup \{v_1\}) = \emptyset$. By Claim 3.1, $v_2 \in D_a$ and v_2 dominates $G_1 - a$, thus $N_{G_1}(v_1) - X = \{a\}$. Consider D_{v_1}, by Fact (2), $D_{v_1} \cap N_{G_1}(v_1) = \emptyset$. Since D_{v_1} dominates a, then $D_{v_1} \cap (N_{G_1}(v_2) - X) \neq \emptyset$. Similarly, $N_{G_1}(v_2) - X$ contains only one vertex, say b. For any vertex $w \in V(G_2) - \{z\}$, $D_w \cap (V(G_1) \cup S)$ has to dominate G_1 and thus $D_w \cap \{v_1, v_2\} = \emptyset$. So we have $a \in D_w$ or $b \in D_w$. Therefore, $D_w \cap V(G_2)$ has to dominate v_1 or v_2, i.e., $D_w \cap (V(G_2) - \{z\}) \neq \emptyset$. Thus, $G_2 - z$ is 2-γ-vertex-critical. By Theorem 1.4, $|V(G)|$ is odd, a contradiction (see Figure 2).

Case 2. $X = \emptyset$.

If $|V(G_1)| = 2$, say $V(G_1) = \{u', v'\}$, then $\deg_G(u') = \deg_G(v') = 2$, a contradiction. Then assume $|V(G_1)| \geqslant 3$. Without loss of generality, let $|N_{G_1}(v_1)| \geqslant 2$. Suppose $x, y \in N_{G_1}(v_1)$, consider D_x, by Fact (2), $v_1 \notin D_x$. Since $v_2y \notin E(G)$, then $v_2 \notin D_x$. Therefore, $D_x \cap V(G_1) \neq \emptyset$, by Claim 3.1, $D_z \cap X \neq \emptyset$, a contradiction.

We complete the proof of Theorem 3.2. □

A graph G is *almost claw-free* (ACF for short) if G is locally 2-dominated and the set of all centers of induced claws is independent. For two nonadjacent vertices u and v in G, let

$$J(u, v) = \{w \in N(u) \cap N(v) \mid N[w] \subseteq N[u] \cup N[v]\}.$$

The vertices of $J(u, v)$ are called the *dominators* of the pair $\{u, v\}$.

A graph G is *quasi-claw-free* (QCF for short) if each pair of vertices $\{u, v\}$ at distance two satisfies the condition $J(u, v) \neq \emptyset$. In particular, every claw-free graph is quasi-claw-free.

A claw $\langle\{v; v_1, v_2, v_3\}\rangle$ is said to be dominated if

$$J(v_1, v_2) \cup J(v_2, v_3) \cup J(v_1, v_3) \neq \emptyset.$$

A graph G is *dominated claw toes*, or a DCT-graph for short, if every claw in G is dominated. Clearly, QCF $\subset$ DCT and also ACF $\subset$ DCT.

The following theorem is proved in [7], here we present a short proof.

Theorem 3.3. *Let G be a $(k+1)$-connected* DCT*-graph, and $|V| \equiv k$* (mod 2)*. Then G is k-factor-critical.*

Proof. Suppose to the contrary that G is not k-factor-critical. Then there exists a minimal subset S with $|S| \geqslant k$ such that $o(G-S) > |S| - k$, by parity, $o(G-S) \geqslant |S| - k + 2 \geqslant 2$. So S is a vertex cut of G. Since G is $(k+1)$-connected, then $|S| \geqslant k+1$ and $o(G-S) \geqslant |S| - k + 2 \geqslant 3$. By the minimality of S, every vertex in S is adjacent to at least three odd components of $G-S$. Let $\langle\{v; v_1, v_2, v_3\}\rangle$ be a claw with claw center $v \in S$ and $v_i \in V(C_i)$. Clearly, $J(v_1, v_2) = \emptyset$, $J(v_2, v_3) = \emptyset$, $J(v_1, v_3) = \emptyset$. Therefore, $J(v_1, v_2) \cup J(v_2, v_3) \cup J(v_1, v_3) = \emptyset$, a contradiction. Therefore, G is k-factor-critical. □

In [3], Ananchuen and Plummer obtained the following result.

Theorem 3.4. *Let G be a connected 3-γ-vertex-critical claw-free graph of even order. Then G is bicritical.*

Combining Theorems 3.2 and 3.3, we can generalize Theorem 3.4 slightly to the following

Theorem 3.5. *Let G be a 3-γ-vertex-critical* DCT*-graph of even order. If the minimum degree is at least three, then G is a brick.*

Acknowledgements This work was partially supported by Major State Basic Research Development Program of China (973 Project) (Grant No. 2006CB805904) and Natural Sciences and Engineering Research Council of Canada (Grant No. 122059–200). The authors cordially thank the support from Center for Combintorics, Nankai University, where the research was carried out.

References

1 Ananchuen N, Plummer M D. Matching properties in domination critical graphs. Discrete Math, 2004, 277: 1–13
2 Ananchuen N, Plummer M D. Matchings in 3-vertex-critical graphs: The even case. Networks, 2005, 45: 210–213
3 Ananchuen N, Plummer M D. On the connectivity and matchings in 3-vertex-critical claw-free graphs. Util Math, 2006, 69: 85–96
4 Ananchuen N, Plummer M D. 3-Factor-criticality in domination critical graphs. Discrete Math, 2007, 307: 3006–3015
5 Brigham R C, Chinn P Z, Dutton R D. Vertex domination-critical graphs. Networks, 1988, 18: 173–179
6 Favaron O. On k-factor-critical graphs. Discuss Math Graph Theory, 1996, 16: 41–51
7 Favaron O, Flandrin E, Ryjáček Z. Factor-criticality and matching extension in *DCT*-graphs. Discuss Math Graph Theory, 1997, 17: 271–278
8 Favaron O, Tian F, Zhang L. Independence and hamiltonicity in 3-domination-critical graphs. J Graph Theory, 1997, 25: 173–184
9 Flandrin E, Tian F, Wei B, Zhang L. Some properties of 3-domination-critical graphs. Discrete Math, 1999, 205: 65–76
10 Fulman J, Hanson D, Macgillivray G. Vertex domination-critical graphs. Networks, 1995, 25: 41–43
11 Lovász L, Plummer M D. Matching Theory. Amsterdam: North-Holland, 1986
12 Tian F, Wei B, Zhang L. Hamiltonicity in 3-domination-critical graphs with $\alpha{=}\delta{+}2$. Discrete Appl Math, 1999, 92: 57–70
13 Wojcicka E. Hamiltonian properties of domination-critical graphs. J Graph Theory, 1990, 14: 205–215
14 Yu Q. Characterizations of various matching extensions in graphs. Australas J Combin, 1993, 7: 55–64