Matching and Factor-Critical Property in 3-Dominating-Critical Graphs*

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Abstract

Let $\gamma(G)$ be the domination number of a graph G. A graph G is *domination-vertex-critical*, or *γ-vertex-critical*, if $\gamma(G - v) < \gamma(G)$ for every vertex $v \in V(G)$. In this paper, we show that: Let G be a γ-vertex-critical graph and $\gamma(G) = 3$. (1) If G is of even order and $K_{1,6}$-free, then G has a perfect matching; (2) If G is of odd order and $K_{1,7}$-free, then G has a near perfect matching with only three exceptions. All these results improve the known results.

Keyword: Vertex coloring, domination number, 3-γ-vertex-critical, matching, near perfect matching, bicritical

MSC: 05C69, 05C70

1 Introduction

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. A set $S \subseteq V$ is a *dominating set* of G if every vertex in V is either in S or is adjacent to a vertex in S. For two sets A and B, A *dominates* B if every vertex of B has a neighbor in A or is a vertex of A; sometimes, we also say that B is *dominated* by A. Let $u \in V$ and $A \subseteq V - \{u\}$, if u is adjacent to some vertex of A, then we say that u is *adjacent* to A. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of dominating sets of G. A graph G is *domination vertex critical*, or *γ-vertex-critical*, if $\gamma(G-v) < \gamma(G)$ for every vertex $v \in V(G)$. Indeed, if $\gamma(G-v) < \gamma(G)$, then $\gamma(G-v) = \gamma(G)-1$. A graph G is *domination edge critical*,

*This work is supported by 973 Project of Ministry of Science and Technology of China and Natural Sciences and Engineering Research Council of Canada.

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if $\gamma(G+e) < \gamma(G)$ for any edge $e \notin E(G)$. We call a graph G *k-γ-vertex-critical* (resp. *k-γ-edge-critical*) if it is domination vertex critical (resp. domination edge critical) and $\gamma(G) = k$.

A matching is *perfect* if it is incident with every vertex of G. If $G - v$ has a perfect matching for every choice of $v \in V(G)$, G is said to be *factor-critical*. The concept of factor-critical graphs was first introduced by Gallai in 1963 and it plays an important role in the study of matching theory. Contrary to its apparent strict condition, such graphs form a relatively rich family for study. It is the essential "building block" for well-known Gallai-Edmonds Matching Structure Theorem.

The subject of γ-vertex-critical graphs was studied first by Brigham, Chinn and Dutton [4] and continued by Fulman, Hanson and MacGillivray [6]. Clearly, the only 1-γ-vertex-critical graph is K_1 (i.e., a single vertex). Brigham, Chinn and Dutton [4] pointed out that the 2-γ-vertex-critical graphs are precisely the family of graphs obtained from the complete graphs K_{2n} with a perfect matching removed (Theorem 1.1). For $k > 2$, however, much remains unknown about the structure of k-γ-vertex-critical graphs. Recently, Ananchuen and Plummer [1, 3] began to investigate matchings in 3-γ-vertex-critical graphs. They showed that a $K_{1,5}$-free 3-γ-vertex-critical graph of *even* order has a perfect matching (see [3]). For the graphs of *odd* order, they proved that the condition of $K_{1,4}$-freedom is sufficient for factor-criticality (see [1]). Wang and Yu [8] improved this result by weakening the condition of $K_{1,4}$-freedom to almost $K_{1,5}$-freedom. In [9], they also studied the k-factor-criticality in 3-γ-edge-critical graphs and obtained several useful results on connectivity of 3-γ-vertex-critical graphs.

The relevant theorems are stated formally below.

Theorem 1.1 (Brigham *et al.*, [4]). *A graph G is 2-γ-vertex-critical if and only if it is isomorphic to K_{2n} with a perfect matching removed.*

Theorem 1.2 (Ananchuen and Plummer, [3]). *Let G be a 3-γ-vertex-critical graph of even order. If G is $K_{1,5}$-free, then G has a perfect matching.*

Theorem 1.3 (Ananchuen and Plummer, [1]). *Let G be a 3-γ-vertex-critical graph of odd order at least 11. If G is $K_{1,5}$-free, then G contains a near perfect matching.*

For $v \in V(G)$, we denote a minimum dominating set of $G - v$ by D_v. The following facts about D_v follow immediately from the definition of 3-γ-vertex-criticality and we shall use it frequently in the proofs of the main theorems.

Facts: If G is 3-γ-vertex-critical, then the followings hold

(1) For every vertex v of G, $|D_v| = 2$;

(2) If $D_v = \{x, y\}$, then x and y are not adjacent to v;

(3) For every pair of distinct vertices v and w, $D_v \neq D_w$.

In this paper, we utilize the techniques developed in [8] and [9] to extend Theorem 1.2 and Theorem 1.3 to the following theorem.

Theorem 1.4. *Let G be a 3-γ-vertex-critical graph.*

(a) *If G is $K_{1,6}$-free and $|V(G)|$ is even, $|V(G)| \neq 12$, then G has a perfect matching.*

(b) *If G is $K_{1,7}$-free of odd order, and $c_o(G) = 1$, $|V(G)| \neq 13$, then either G has a near perfect matching or G is one of Fig. 1 and Fig. 4.*

In theory of matching, Tutte's 1-Factor Theorem plays a central role. From 1-Factor Theorem, a characterization of a graph with a near perfect matching can be easily derived. Following the convention of [7], we use $c(G)$ (resp. $c_o(G)$) to denote the number of (resp. odd) components of G.

Theorem 1.5 (Tutte's 1-Factor Theorem)**.** *A graph G has a perfect matching if and only if for any $S \subseteq V(G)$, $c_o(G - S) \leqslant |S|$.*

Theorem 1.6. *A graph G of odd order has no near perfect matching if and only if there exits a set $S \subseteq V(G)$, $c_o(G - S) \geqslant |S| + 3$.*

Proof. Let G' be a graph obtained from G by adding a new vertex u and joining u to every vertex of G. Then G has a near perfect matching if and only if G' has a perfect matching.

By Tutte's 1-Factor Theroem, and the parity, G' has no perfect matching if and only if there exists a vertex set $S' \subseteq V(G')$ such that $c_o(G' - S') \geqslant |S'| + 2$. Since u is adjacent to every vertex of G, then $u \in S'$. Let $S = S' \setminus \{u\} \subseteq V(G)$. Then $c_o(G - S) = c_o(G' - S') \geqslant |S'| + 2 = |S| + 3$. ∎

The following lemma is proven by Ananchuen and Plummer in [1], they are useful to deal with the graphs with smaller cut sets. We will use them in our proof several times.

Lemma 1. *Let G be a 3-γ-vertex-critical graph.*

(a) *If G is disconnected, then $G = 3K_1$ or G is a disjoint union of a 2-γ-vertex-critical graph and an isolated vertex;*

(b) *If G has a cut-vertex u, then $c(G - u) = 2$. Furthermore, let C_i be a component of $G - u$ ($i = 1, 2$), then $G[V(C_i) \cup \{u\}]$ is 2-γ-vertex-critical;*

(c) *If G has a 2-cut S, then $c(G - S) \leqslant 3$. Furthermore, if $c(G - S) = 3$, then $G - S$ must contain at least one singleton.*

We also need the following results in our proof.

Lemma 2 (Wang and Yu, [8])**.** *Let G be a 3-γ-vertex-critical graph and $S \subseteq V(G)$. If $D_u \subseteq S$ for each vertex $u \in S$, then there exists no vertex of degree one in $G[S]$.*

Theorem 1.7 (Wang and Yu, [9])**.** *Let G be a 3-γ-vertex-critical graph of even order. If the minimum degree is at least three, then G is 3-connected.*

Theorem 1.8 (Mantel, see [10])**.** *The maximum number of edges in a triangle-free simple graph of order n is $\lfloor \frac{n^2}{4} \rfloor$.*

2 Proof of Theorem 1.4

In this section, we provide a proof of Theorem 1.4.

Proof. Suppose, to the contrary, that the theorem does not hold. From Theorem 1.5 and Theorem 1.6, and the parity, there exists a vertex set $S \subseteq V(G)$, such that $c_o(G-S) \geqslant |S|+k-4$ ($k=6,7$). Without loss of generality, let S be *minimal* such a set. By Lemma 1, $|S| \geqslant 3$.

Claim 1. *Each vertex of S is adjacent to at least three odd components of $G-S$.*

Otherwise, there exists a vertex $v \in S$ such that v is adjacent to at most two odd components of $G-S$. Let $S' = S - \{v\}$. It is easy to see that S' is a nonempty set which satisfies the condition $c_o(G-S') \geqslant |S'|+k-4$, contradicting the minimality of S.

Let $C_1, C_2, \ldots, C_t$ be the odd components and $E_1, E_2, \ldots, E_n$ be the even components of $G-S$.

Case 1. $|S| = 3$, say $S = \{u, v, w\}$.
Then $t \geqslant k-1$.

Claim 2. *For every vertex $s \in S$, $D_s \subseteq S$.*

Clearly, $D_s \cap S \neq \emptyset$. Assume $D_v = \{u, v'\}$, where $v' \in V(C_1 \cup E_1)$. This means that, if the vertex v' is in the odd component of $G-S$, we assume $v' \in V(C_1)$; if it is in the even component of $G-S$, we assume $v' \in V(E_1)$. By Fact 2, $vu \notin E(G)$, $vv' \notin E(G)$, and u dominates $C_2 \cup C_3 \cup \cdots \cup C_t$. By Claim 1, w is adjacent to at least two of $C_2, C_3, \ldots, C_t$. Without loss of generality, let $wc_i \in E(G)$, for some $c_i \in V(C_i)$, $i = 2, 3$. By Fact 2 again, $D_{c_i} \cap S = \{v\}$, $i = 2, 3$. Then $vc_i \notin E(G)$. Since $vv' \notin E(G)$, then $D_{c_2} \cap V(C_1 \cup E_1) \neq \emptyset$. But D_{c_2} can not dominate c_3, a contradiction. The claim is proved.

By Claim 2 and Fact 2, S is an independent set, and for any vertex $x \notin S$, $|N_S(x)| \geqslant 2$. In fact, $|N_S(x)| = 2$. Since, if $|N_S(x)| = 3$, then $D_x \cap S = \emptyset$.

Claim 3. *If $t \geqslant 5$, then $G-S$ has no even component.*

Suppose, to the contrary, that there exists an even component E_1. Choose a vertex $x \in V(E_1)$, and consider D_x. Assume $D_x = \{u, u'\}$, where $u \in S$ and u' is in C_1 or in an even component. Then u dominates $C_2 \cup C_3 \cup \cdots \cup C_t$. By Claim 1, w is adjacent to at least two of $C_2, C_3, \ldots, C_t$. Without loss of generality, let $wc_i \in E(G)$, where $c_i \in V(C_i)$, $i = 2, 3$. By Fact 2, $D_{c_i} \cap S = \{v\}$, thus $vc_i \notin E(G)$ for $i = 2, 3$. Then $D_{c_2} \cap V(C_3) \neq \emptyset$ and v dominates $C_1 \cup C_4 \cup C_5 \cup E_1$. Henceforth $D_{c_j} \cap S = \{w\}$ and $wc_j \notin E(G)$, $j = 4, 5$. Consider D_{c_4}, since $wc_5 \notin E(G)$, then $D_{c_4} \cap V(C_5) \neq \emptyset$ and hence w dominates $C_1 \cup C_2 \cup C_3$ and E_1. Since every vertex of C_1 is adjacent to both w and v, then u is not adjacent to any vertex of

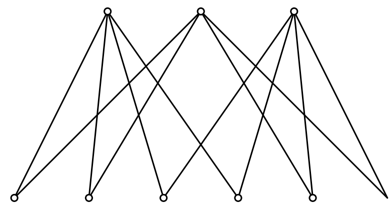

Fig. 1: A 9-vertex graph which has no near perfect matching.

C_1, hence $u' \in V(C_1)$. Since $\{u, u'\}$ dominates $G - \{x\}$, then u dominates $E_1 - \{x\}$. Since $|E_1| \geqslant 2$, then every vertex of $V(E_1) - \{x\}$ is adjacent to every vertex of S, a contradiction. So $G - S$ has no even component.

Case 1.1. There exists a (odd) component, say C_1, and a vertex $c \in V(C_1)$ such that $D_c \cap V(C_1) \neq \emptyset$.

Let $D_c = \{u, c'\}$, where $c' \in V(C_1)$. Then u dominates $C_2 \cup C_3 \cup \cdots \cup C_t$. Let $c_i \in V(C_i)$, $i = 2, \ldots, t$. Since $|N_S(c_i)| = 2$ and $uc_i \in E(G)$, assume $wc_2 \in E(G)$ and $wc_3 \in E(G)$. Then $D_{c_i} \cap S = \{v\}$ and $vc_i \notin E(G)$ for $i = 2, 3$. Since $vc_3 \notin E(G)$, then $D_{c_2} \cap V(C_3) \neq \emptyset$. Therefore, v dominates $C_1 \cup C_4 \cup C_5$, and hence $wc_4 \notin E(G)$ and $wc_5 \notin E(G)$. Then w dominates $C_1 \cup C_2 \cup C_3$. So every vertex of C_1 is adjacent to both w and v, then u is not adjacent to any vertex of C_1. Therefore, for any vertex $x \in V(C_1)$, $D_x \cap S = \{u\}$ and $|D_x \cap V(C_1)| = 1$. It is easy to see that C_1 is 2-γ-vertex-critical, and thus $|V(C_1)|$ is even, a contradiction.

Case 1.2. For any vertex x of C_i, $D_x \cap V(C_i) = \emptyset$.

Assume that $|V(C_1)| \geqslant 3$. Let $x \in V(C_1)$, $D_x = \{u, x'\}$. By Claim 2 and the assumption $D_x \cap V(C_1) = \emptyset$, we may assume that $x' \in V(C_2)$. Then u dominates $C_3 \cup C_4 \cup C_5$ and $C_1 - \{x\}$. Since $|N_S(c_i)| = 2$ and $uc_i \in E(G)$ for $i = 3, 4, 5$, so we assume $wc_3 \in E(G)$ and $wc_4 \in E(G)$. Then $vc_3 \notin E(G)$ and $vc_4 \notin E(G)$. So $D_{c_3} \cap V(C_4) \neq \emptyset$. It yields that v dominates C_1. Since every vertex of $V(C_1) - \{x\}$ is adjacent to both u and v, then it is not adjacent to w. Let $y \in V(C_1) - \{x\}$. Then $D_y \cap S = \{w\}$, by the assumption $D_y \cap V(C_1) = \emptyset$, so D_y can not dominate $V(C_1) - \{x, y\}$, a contradiction.

Therefore all the components of $G - S$ are singletons, i.e., $C_i = \{c_i\}$. Assume $D_{c_1} = \{u, c_2\}$. Then $uc_1 \notin E(G)$, $c_2v \in E(G)$ and $c_2w \in E(G)$. Since $|N_S(c_2)| = 2$, then $c_2u \notin E(G)$. Thus u dominates $G - S - \{c_1, c_2\}$. Therefore, $D_{c_2} = \{u, c_1\}$. Similarly, we see $D_{c_3} = \{v, c_4\}$, $D_{c_4} = \{v, c_3\}$, $D_{c_5} = \{w, c_6\}$ and $D_{c_6} = \{w, c_5\}$. Hence, there is only one 9-vertex graph satisfying these conditions (see Fig. 1).

Case 2. $|S| = 4$, and thus $t \geqslant k$.

We first show that there exists a vertex $a \in S$ such that $D_a \nsubseteq S$. Otherwise, $D_b \subseteq S$ for every vertex $b \in S$. By Fact 2 and Lemma 2, S is an independent set.

It is easy to check that this is impossible.

So let u be a vertex of S with $D_u \nsubseteq S$. Clearly, $D_u \cap S \neq \emptyset$. Let $D_u = \{v, x\}$, where $v \in S$ and $x \in V(G) - S$. Since G is $K_{1,k}$-free, so $t = k$ and $G - S$ has no even component. Without loss of generality, let $x \in V(C_1)$, then v dominates all vertices of $\bigcup_{i=2}^{k} V(C_i)$. Moreover, by $K_{1,k}$-freedom again, $C_2, C_3, \ldots, C_k$ are all complete, and v is not adjacent to any vertex of $V(C_1)$.

Let $S - \{u, v\} = \{w, z\}$. By Claim 1, let $wc_i \in E(G)$, for some $c_i \in V(C_i)$, $i = 2, 3$. Then $z \in D_{c_2}$. Otherwise, we have $D_{c_2} \cap S = \{u\}$. Since $ux \notin E(G)$, then $D_{c_2} \cap V(C_1) \neq \emptyset$, but then D_{c_2} can not dominate v, a contradiction. Similarly, $z \in D_{c_3}$, thus $zc_2 \notin E(G)$ and $zc_3 \notin E(G)$. By Facts 2 and 3, either $D_{c_2} \neq \{u, z\}$ or $D_{c_3} \neq \{u, z\}$. Assume that $D_{c_2} \neq \{u, z\}$, thus $D_{c_2} \cap S = \{z\}$. Since $zc_3 \notin E(G)$, then $D_{c_2} \cap V(C_3) \neq \emptyset$, and z dominates $V(C_1) \cup V(C_4) \cup V(C_5) \cup V(C_6)$. By a similar argument, $w \in D_{c_j}$, for some $c_j \in V(C_j)$, $j = 4, 5, 6$. Furthermore, $wc_j \notin E(G)$, $j = 4, 5, 6$. From Fact 3, $D_{c_4} \neq \{u, w\}$ or $D_{c_5} \neq \{u, w\}$ or $D_{c_6} \neq \{u, w\}$. Assume $D_{c_4} \neq \{u, w\}$. Since $wc_5 \notin E(G)$, then $D_{c_4} \cap V(C_5) \neq \emptyset$, but D_{c_4} can not dominate c_6, a contradiction.

Case 3. $|S| = 5$, and thus $t \geqslant k + 1$.

Claim 4. *For every vertex $s \in S$, $D_s \subseteq S$.*

Otherwise, $D_u \nsubseteq S$ for some $u \in S$. Clearly, $D_u \cap S \neq \emptyset$. Let $D_u = \{y, z\}$, where $y \in S$ and $z \notin S$. Since $t \geqslant k+1$, y must dominate at least k odd components of $G - S$, which contradicts to $K_{1,k}$-freedom.

By Claim 4 and Lemma 2, each vertex of S has degree 0 or 2 in $G[S]$. It is not hard to see that $G[S]$ can only be a 5-cycle or a disjoint union of a 4-cycle and an isolated vertex. Let $S = \{s_1, s_2, s_3, s_4, s_5\}$. There are $\binom{5}{2} = 10$ distinct pairs of vertices in S. By Fact 3 and Claim 4, there must exist a vertex x in an odd component of $G - S$ such that $D_x \nsubseteq S$. Assume that $x \in V(C_1)$. Clearly, $D_x \cap S \neq \emptyset$. Since G is $K_{1,k}$-free, we have $t = k + 1$ and $G - S$ has no even component.

Case 3.1. $G[S]$ is a 5-cycle.

Let $s_1 s_2 s_3 s_4 s_5 s_1$ be the 5-cycle in the counterclockwise order and $D_x = \{s_1, x'\}$, where $x' \notin S$. Since G is $K_{1,k}$-free, then $x' \notin V(C_1)$. Assume that $x' \in V(C_2)$. Then s_1 dominates $\bigcup_{i=3}^{k+1} V(C_i)$ and x' is adjacent to both s_3 and s_4. Moreover, $K_{1,k}$-freedom of G implies that $C_3, C_4, \ldots, C_{k+1}$ are all complete and s_1 is not adjacent to any vertex of $V(C_1) \cup V(C_2)$. Henceforth, C_1 is a singleton (i.e., $V(C_1) = \{x\}$).

Since $D_{s_3} = \{s_1, s_5\}$, then s_5 dominates $V(C_1) \cup V(C_2)$. Similarly, since $D_{s_4} = \{s_1, s_2\}$, s_2 dominates $V(C_1) \cup V(C_2)$. Therefore, x' is adjacent to all vertices of $S - \{s_1\}$. Now consider $D_{x'}$. Since $D_{x'} \cap S = \{s_1\}$ and $s_1 x \notin E(G)$, it follows that $D_{x'} = \{s_1, x\}$. Hence, x is adjacent to both s_3 and s_4, and $V(C_2) = \{x'\}$. But then $\{s_1, s_3\}$ is a dominating set in G, a contradiction to $\gamma(G) = 3$.

Case 3.2. $G[S]$ is a disjoint union of a 4-cycle and an isolated vertex.

Let $s_1s_2s_3s_4s_1$ be the 4-cycle in the counterclockwise order and s_5 be the isolated vertex in $G[S]$. Then $D_{s_1} = \{s_3, s_5\}$, $D_{s_2} = \{s_4, s_5\}$, $D_{s_3} = \{s_1, s_5\}$, and $D_{s_4} = \{s_2, s_5\}$.

Since G is $K_{1,k}$-free, s_5 is adjacent to at most $k-1$ (odd) components of $G - S$. Without loss of generality, let $C_1, \ldots, C_r$ be the components which are not adjacent to s_5. Then $t = k+1$ implies $r \geqslant 2$. Thus s_i dominates $\bigcup_{j=1}^{r} V(C_j)$ for $i = 1, 2, 3, 4$. Now consider D_{c_1}, where $c_1 \in V(C_1)$. Clearly, $D_{c_1} \cap S = \{s_5\}$. Since s_5 is not adjacent to $V(C_2)$, then $D_{c_1} \cap V(C_2) \neq \emptyset$. Therefore, $r = 2$ and s_5 dominates $\bigcup_{j=3}^{k+1} V(C_j)$. Moreover, $V(C_1) = \{c_1\}$. By a similar argument, C_2 is also a singleton.

For any vertex $v \in \bigcup_{j=3}^{k+1} V(C_j)$, by Fact 2, $s_5 \notin D_v$, but the vertices in $S - \{s_5\}$ do not dominate s_5. Then $D_v \nsubseteq S$ and $D_v \cap \{s_1, s_2, s_3, s_4\} \neq \emptyset$. From $K_{1,k}$-freedom of G, it implies that $C_3, C_4, \ldots, C_{k+1}$ are all singletons, say $V(C_j) = \{c_j\}$ for $j = 3, \ldots, k+1$. Then $|V(G)| = 12$ or 13 (see examples: Fig. 2, Fig. 3).

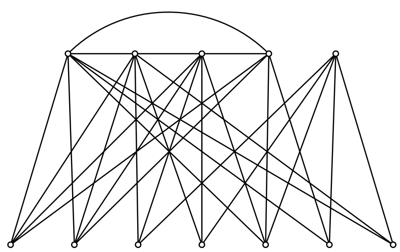

Fig. 2: A $K_{1,6}$-free graph without perfect matching.

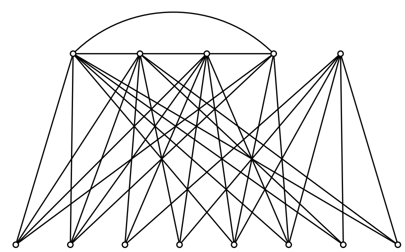

Fig. 3: A $K_{1,7}$-free graph without near perfect matching.

Case 4. $|S| \geqslant 6$, and thus $t \geqslant k+2$.

Claim 5. *For every vertex $s \in V(G)$, $D_s \subseteq S$.*

Suppose that $D_x \nsubseteq S$ for some $x \in V(G)$. Clearly, $D_x \cap S \neq \emptyset$. Let $D_x = \{y, z\}$, where $y \in S$ and $z \notin S$. Since $t \geqslant k+2$, y must dominate at least k odd components of $G - S$, a contradiction.

For each $i = 1, \ldots, t$, let $S_i \subseteq S$ be the set of vertices in S which are adjacent to some vertex in C_i, and let $d = \min\{|S_i|\}$. Without loss of generality, assume that $|S_1| = d$. Note that for any vertex $v \in V(G) - V(C_1)$, $D_v \subset S$ has to dominate C_1, thus, $D_v \cap S_1 \neq \emptyset$. We call such a set D_v *normal 2-set associated with v and S_1*, or *normal set* in short. By a simple counting argument, we see that there are at most $\binom{|S|}{2} - \binom{|S|-d}{2}$ normal sets.

Case 4.1. G is $K_{1,6}$-free, and $|V(G)|$ is even.

Since every vertex in S is adjacent to at most five components of $G-S$, then $c(G-S)\leqslant 10$. Henceforth, $6\leqslant |S|\leqslant 8$ and $d\leqslant \lfloor\frac{5|S|}{|S|+2}\rfloor\leqslant 4$.

If $|S|=6$, then $\binom{6}{2}-\binom{6-d}{2}\geqslant 13$, and thus $d\geqslant 4$. But $d\leqslant\lfloor\frac{5\times 6}{6+2}\rfloor<4$, a contradiction.

If $|S|=7$, then

$$\binom{7}{2}-\binom{7-d}{2}\geqslant 15 \tag{2.1}$$

or $d\geqslant 3$. Since $d\leqslant\lfloor\frac{5\times 7}{7+2}\rfloor<4$, then $d=3$ and the equality holds in (2.1). Let $S_1=\{u,v,w\}$, then $\{u,v\}$, $\{u,w\}$, $\{v,w\}$ are all corresponding to some D_x where $x\notin V(C_1)$. Since u is adjacent to at most five components of $G-S$, so we may assume that u is not adjacent to $C_6,C_7,\ldots,C_9$. Then v dominates at least three of them, and v is adjacent to at most two of $C_1,C_2,\ldots,C_5$. Similarly, w is adjacent to at most two of $C_1,C_2,\ldots,C_5$. Both v and w are adjacent to C_1, then $\{v,w\}$ can dominate at most two of $C_2,C_3,\ldots,C_5$, hence it can not be realized a D_x for some $x\notin V(C_1)$, a contradiction.

If $|S|=8$, then $c(G-S)=c_o(G-S)=10$. We construct a graph H with vertex set S and $uv\in E(H)$ if and only if $D_x=\{u,v\}$ for some $x\in V(G)$. We show that H is triangle-free. Let $u,v,w\in S$, if $uv\in E(H)$, $uw\in E(H)$ and u is not adjacent to $C_6,\ldots,C_{10}$, then both v and w are adjacent to at least four of them. Hence both v and w are adjacent to at most one component of $C_1,C_2,\ldots,C_5$. Therefore $\{v,w\}$ is not a D_x for any $x\in V(G)$. By Theorem 1.8, $|E(H)|\leqslant\lfloor\frac{8^2}{4}\rfloor=16<|V(G)|$, a contradiction.

Case 4.2. G is $K_{1,7}$-free and $|V(G)|$ is odd.

Since every vertex in S is adjacent to at most six components of $G-S$, then $c(G-S)\leqslant 12$. So $6\leqslant |S|\leqslant 9$.

If $|S|=6$, by Claim 5 and Fact 3, $\binom{6}{2}\geqslant |V(G)|\geqslant 6+9$. Then $|V(G)|=15$, and $G-S$ is an independent set of nine vertices. Moreover, every pair in S is corresponding to a D_x for some $x\in V(G)$. As $\binom{6}{2}-\binom{6-d}{2}\geqslant 14$, so $d\geqslant 4$. For any $x\notin S$, $D_x\subset S$, by Fact 2, every vertex in $G-S$ has degree 4, and then every vertex of S is adjacent to six components of $G-S$. Let δ be the minimum degree of $G[S]$ and $d_{G[S]}(u)=d$. If $d\leq 2$, then there exists at least one pair in $S\setminus N_{G[S]}[u]$ which is not corresponding to D_u, and thus it does not dominate u, a contradiction. By Fact 2, $G[S]$ is a 3-regular graph. From the above information, it is not hard to see that there are only two such graphs (see Fig. 4).

If $|S|=7$, we construct an auxiliary graph H with vertex set S and $uv\in E(H)$ if and only if $D_x=\{u,v\}$ for some $x\in S$. Assume that $uv,uw\in E(H)$, and u is not adjacent to $C_7,\ldots,C_{10}$. Then both v and w dominate $C_7,\ldots,C_{10}$, and are all adjacent to at most two of $C_1,C_2,\ldots,C_6$. Hence $\{v,w\}$ can not be realized as a D_v for some $v\in V(G)$. Therefore, H is triangle-free. If H contains a cycle of length at least five, then at least five pairs can not be realized as a D_x for some $x\in V(G)$, $\binom{7}{2}-5=16<17\leqslant|V(G)|$, a contradiction. As $|E(H)|>|V(H)|-1$, so H only

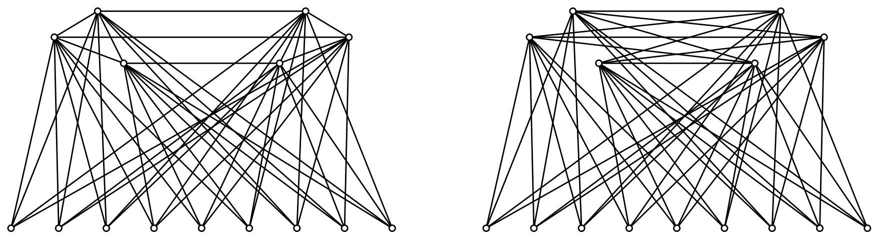

Fig. 4: Two exceptions when G is $K_{1,7}$-free and $|V(G)|$ is odd.

contains cycles of length four, and H is bipartite. Let $s_1s_2s_3s_4$ be a four cycle in H. $|E(H)| > |V(H)| - 1 = 6$, it yields that the component which contains the 4-cycle $s_1s_2s_3s_4$, say H', has at least six vertices. The pairs in the same partite of H' can not be realized as a D_x for some $x \in V(G)$, a simple counting argument shows that H has at least five such pairs. So $\binom{7}{2} - 5 = 16 < 17 \leqslant |V(G)|$, a contradiction.

If $8 \leqslant |S| \leqslant 9$, we construct a graph H as in the case that "G is $K_{1,6}$-free, $|V(G)|$ is even, and $|S| = 8$". Similarly, H is triangle-free, by Theorem 1.8, $|E(G)| \leqslant \lfloor \frac{|S|^2}{4} \rfloor < |V(G)|$, a contradiction. ∎

Remark 1. The conclusion in this theorem holds for all graphs except $|V(G)| = 12$ or 13. For these cases, we can determine the exceptions precisely in some cases (such as in Case 4.2) but fail to determine all of them in other cases (such as in Case 3.2). With some efforts, one may be able to find all graphs which have no perfect matching or near-perfect matching for $|V(G)| = 12$ or 13.

Remark 2. Ananchuen and Plummer [2] showed that: let G be a connected 3-γ-vertex-critical graph of even order. If G is claw-free, then G is bicritical. The authors also generalized this result, and proved that: let G be a 3-γ-vertex-critical graph of even order, if G is $K_{1,4}$-free, and the minimum degree is at least four, then G is bicritical. This result will be published in a future article.

Acknowledgment The authors would like to thank the anonymous referee and the editor for their valuable comments and suggestions.

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