Vertex-coloring 2-edge-weighting of graphs

Hongliang Lu\textsuperscript{a}, Qinglin Yu\textsuperscript{b,1}, Cun-Quan Zhang\textsuperscript{c}

\textsuperscript{a} Department of Mathematics, Xi’an Jiaotong University, Xi’an, China
\textsuperscript{b} Department of Mathematics and Statistics, Thompson Rivers University, Kamloops, BC, Canada
\textsuperscript{c} Department of Mathematics, West Virginia University, Morgantown, WV, USA

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\textbf{A B S T R A C T}

A \textit{k}-edge-weighting \( w \) of a graph \( G \) is an assignment of an integer weight, \( w(e) \in \{1, \ldots, k\} \), to each edge \( e \). An edge weighting naturally induces a vertex coloring \( c \) by defining \( c(u) = \sum_{v \sim e} w(e) \) for every \( u \in V(G) \). A \textit{k}-edge-weighting of a graph \( G \) is vertex-coloring if the induced coloring \( c \) is proper, i.e., \( c(u) \neq c(v) \) for any edge \( uv \in E(G) \).

Given a graph \( G \) and a vertex coloring \( c_0 \), does there exist an edge-weighting such that the induced vertex coloring is \( c_0 \)? We investigate this problem by considering edge-weightings defined on an abelian group.

It was proved that every 3-colorable graph admits a vertex-coloring 3-edge-weighting (Karoński et al. (2004) \cite{12}). Does every 2-colorable graph (i.e., bipartite graphs) admit a vertex-coloring 2-edge-weighting? We obtain several simple sufficient conditions for graphs to be vertex-coloring 2-edge-weighting. In particular, we show that 3-connected bipartite graphs admit vertex-coloring 2-edge-weighting.

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\textbf{1. Introduction}

In this paper, we consider only finite, undirected and simple connected graphs. For a vertex \( v \) of a graph \( G = (V, E) \), \( N_G(v) \) denotes the set of vertices which are adjacent to \( v \). If \( v \in V(G) \) and \( e \in E(G) \), we use \( v \sim e \) to denote that \( v \) is an end-vertex of \( e \). \( \omega(G) \) denotes the number of connected components of \( G \). An \textit{k}-vertex coloring \( c \) of \( G \) is an assignment of \( k \) integers, \( 1, 2, \ldots, k \), to the vertices

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\textsuperscript{1} Tel.: +1 250 3715552; fax: +1 250 3715675.

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of \( G \), the color of a vertex \( v \) is denoted by \( c(v) \). The coloring is proper if no two distinct adjacent vertices share the same color. A graph \( G \) is \( k \)-colorable if \( G \) has a proper \( k \)-vertex coloring. The chromatic number \( \chi(G) \) is the minimum number \( r \) such that \( G \) is \( r \)-colorable. Notation and terminology that is not defined here may be found in [6].

A \( k \)-edge-weighting \( w \) of a graph \( G \) is an assignment of an integer weight \( w(e) \in \{1, \ldots, k\} \) to each edge \( e \) of \( G \). An edge weighting naturally induces a vertex coloring \( c(u) \) by defining \( c(u) = \sum_{u \sim e} w(e) \) for every \( u \in V(G) \). An \( k \)-edge-weighting of a graph \( G \) is vertex-coloring if for every edge \( e = uv, c(u) \neq c(v) \) and then we say \( G \) admitting a vertex-coloring \( k \)-edge-weighting. There are many variations of vertex-coloring edge-weighting, for instance, a graph \( G \) is vertex-injective if for any pair of vertices \( u, v \), \( c(u) \neq c(v) \). Another concept is irregularity strength, which is a different approach but is similar enough. A multigraph is irregular if no two vertex degrees are equal. A multigraph can be viewed as a weighted graph with nonnegative integer weights on the edges. The degree of a vertex in a weighted graph is the sum of the incident weights. Chartrand et al. [9] defined the irregularity strength of a graph \( G \) to be the minimum of the maximum edge weight in an irregular multigraph with underlying graph \( G \). Other related results and variations can be found in [1,4,5,7] and [10].

If a graph has an edge as a component, clearly it cannot have a vertex-coloring \( k \)-edge-weighting. So in this paper, we only consider graphs without a \( K_2 \) component and refer to such graphs as nice graphs.

In [12], Karoński, Łuczak and Thomason initiated the study of vertex-coloring \( k \)-edge-weighting and they brought forward a conjecture as following.

Conjecture 1.1 (1–2–3-Conjecture). Every nice graph admits a vertex-coloring 3-edge-weighting.

Furthermore, they proved that the conjecture holds for 3-colorable graphs (see Theorem 1 in [12]). For other graphs, Addario-Berry et al. [2] showed that every nice graph admits a vertex-coloring 30-edge-weighting. Addario-Berry et al. [3] improved the number of integers required to 16. Later, Wang and Yu [13] improved this bound to 13. Recently, Kalkowski et al. [11] showed that every nice graph admits a vertex-coloring 5-edge-weighting, which is a great leap towards the 1–2–3-Conjecture.

In this paper, we focus on vertex-coloring 2-edge-weighting. In Section 2, we present several new results about vertex-coloring 2-edge-weighting.

Besides the existence problem of vertex-coloring \( k \)-edge-weighting, a natural question to ask is that given a graph \( G \) and a vertex coloring \( c_0 \), can we realize the coloring \( c_0 \) by a \( k \)-edge-weighting, i.e., does there exist an edge-weighting such that the induced vertex coloring is \( c_0 \)? For general graphs, it is not easy to find such an edge-weighting. However, by restricting edge weights to an abelian group, we obtain a neat positive answer for this even for a non-proper coloring \( c_0 \). In Section 3, we show that every 3-connected nice bipartite graph admits a vertex-coloring 2-edge-weighting.

2. Vertex-coloring 2-edge-weighting

For a graph \( G \), there is a close relationship between 2-edge-weightings and graph factors. Namely, a 2-edge-weighting problem is equivalent to finding a special factor of graphs (see [2,3]). So to find spanning subgraphs with pre-specified degree is an important part of edge-weighting. We shall use some of these results in our proofs.

Lemma 2.1 ([3]). Given a graph \( G = (V, E) \), if for all \( v \in V \), there are integers \( a_v^-, a_v^+ \) such that \( a_v^- \leq \frac{1}{2}d(v) \leq a_v^+ < d(v) \), and

\[
a_v^+ \leq \min \left\{ \frac{1}{2} (d(v) + a_v^-) + 1, 2(a_v^- + 1) + 1 \right\},
\]

then there exists a spanning subgraph \( H \) of \( G \) such that \( d_H(v) \in \{a_v^- + 1, a_v^+, a_v^+ + 1\} \).

Given an arbitrary vertex coloring \( c_0 \), we want to find an edge-weighting such that the induced vertex coloring is \( c_0 \)? Under a weak condition, the next two theorems show that there exists an edge-weighting from an abelian group to \( E(G) \) to induce \( c_0 \) for bipartite and non-bipartite graphs respectively.
Theorem 2.2. Let $G$ be a non-bipartite graph and $\Gamma = \{g_1, g_2, \ldots, g_k\}$ be a finite abelian group, where $k = |\Gamma|$. Let $c_0$ be any $k$-vertex coloring of $G$ with color classes \{U_1, \ldots, U_k\}, where $|U_i| = n_i (1 \leq i \leq k)$. If there exists an element $h \in \Gamma$ such that $n_1 g_1 + \cdots + n_k g_k = 2h$, then there is an edge-weighting with the elements of $\Gamma$ such that the induced vertex coloring is $c_0$.

Proof. Let $c_0$ be any $k$-vertex coloring with vertex partition \{U_1, \ldots, U_k\}, where every element in $U_i$ is colored with $g_i (1 \leq i \leq k)$ such that $n_1 g_1 + \cdots + n_k g_k = 2h$.

Assign one edge with weight $h$ and the rest with zero, so the sum of vertex colors is $2h$. We now adjust this initial weighting, while maintaining the sum of vertex weights, until all the vertices in $U_i$ have color $g_i (1 \leq i \leq k)$. Suppose there exists a vertex $v \in U_i$ with the wrong color $g \neq g_i$. Since $n_1 g_1 + \cdots + n_k g_k = 2h$, there must be another vertex $u \in V(G)$ whose color is also wrong. Since $G$ is non-bipartite, we can choose a walk of even length from $u$ to $v$, which is always possible since $k \geq 3$. Traverse this walk, adding $g_i - g, g - g_i, g_i - g, \ldots$ alternately to the edges as they are encountered. This operation maintains the sum of vertex weights, leaves the colors of all but $u$ and $v$ unchanged, and yields one more vertex of the correct color. Hence, repeated applications give the desired weighting. □

Note that in Theorem 2.2, the given vertex-coloring $c_0$ can be either a proper or an improper coloring.

Theorem 2.3. Let $G$ be a nice bipartite graph and $Z_2 = \{0, 1\}$. Let $c_0$ be any 2-vertex coloring of $G$ with color classes \{U_0, U_1\}, where $|U_i| = n_i$ with $c_0(U_i) = i (i = 0, 1)$. If $n_1$ is even, then there exists an edge-weighting with the elements of $Z_2$ such that the induced vertex coloring is $c_0$.

Proof. Let $g_1 = 0$ and $g_2 = 1$. If there is a vertex $u$ of color $g_i$ with the wrong color $g \neq g_i$, and since $n_2$ is even, then there must be another vertex $v \in V(G)$ whose color is also wrong. Since $G$ is connected, then there is a path from $u$ to $v$. Traverse this walk and add $1, 1, 1, \ldots$ to the edges as they are encountered. This operation always maintains the sum of vertex colors, leaves the colors of all but $u$ and $v$ unchanged, and yields one more vertex of the correct weight. □

Remark. The edge-weighting problem on groups has been studied by Karoński et al. in [12]. They proved that for each $|\Gamma|$-colorable graph $G$, there exists an edge-weighting with the elements of $\Gamma$ such that the induced vertex-coloring is proper. Our proofs of Theorems 2.2 and 2.3 are modifications of that result.

For the convenience of applying Theorem 2.3, we restate it in terms of 1, 2 as follows.

Corollary 2.4. Let $G$ be a nice bipartite graph. Let $U \subseteq V(G)$ and $\overline{U} = V(G) - U$, where $|U| = n_1$ and $|\overline{U}| = n_2$. If $n_1$ is even, then there exists an edge-weighting with the elements from $\{1, 2\}$ such that the induced vertex coloring $c$ satisfies that $c(x)$ is odd for all $x \in U$ and $c(y)$ is even for all $y \in \overline{U}$.

Proof. Let $c_0 : V(G) \rightarrow \{0, 1\}$ such that $c_0(U) = \{1\}$ and $c_0(\overline{U}) = \{0\}$. By Theorem 2.3, there exists an edge-weighting, say $w$, with the elements of $Z_2$ such that the induced vertex coloring is $c_0$. Let $w' : E(G) \rightarrow \{1, 2\}$ be defined as follows:

$$w'(e) = \begin{cases} 2 & \text{if } w(e) = 0, \\ 1 & \text{if } w(e) = 1. \end{cases}$$

Then $w'$ is a desired edge-weighting. □

It was proved in [12] that every 3-colorable graph has a vertex-coloring 3-edge-weighting. A natural question to ask is whether every 2-colorable graph (i.e., bipartite graphs) has a vertex-coloring 2-edge-weighting. But the answer is no, since $C_6$ and $C_{10}$ do not admit vertex-coloring 2-edge-weightings. In fact, Chang et al. [8] proved the following results.

Lemma 2.5 ([8]). Every connected nice bipartite graph admits a vertex-coloring 2-edge-weighting if one of the following conditions holds:

1. $|A|$ or $|B|$ is even;
2. $\delta(G) = 1$;
3. $|d(u)/2| + 1 \neq d(v)$ for any edge $uv \in E(G)$. 

An interesting corollary of Lemma 2.5 is that every \( r \)-regular nice bipartite graph \((r \geq 3)\) admits a vertex-coloring 2-edge-weighting. More generally, every bipartite \([r, r + 1]\)-graph \(G\) (i.e., \(d_G(v) \in [r, r + 1]\) for any \(v \in V(G)\)) with \(r \geq 4\) admits a vertex-coloring 2-edge-weighting.

**Theorem 2.6.** Let \(G\) be a nice graph. If \(\delta(G) \geq 8\chi(G)\), then \(G\) admits a vertex-coloring 2-edge-weighting.

**Proof.** Let \(\{V_1, \ldots, V_{\chi(G)}\}\) be a partition of \(V(G)\) into independent sets. For each \(v \in V_i\), choose \(a_v^-\) such that \(\left\lfloor \frac{d(v)}{4} \right\rfloor \leq a_v^- \leq \left\lceil \frac{d(v)}{2} \right\rceil\), \(a_v^- + d_G(v) \equiv 2i \pmod{2\chi(G)}\), and \(a_v^- + 2\chi(G) \geq \left\lfloor \frac{d(v)}{2} \right\rfloor\). Such choice for \(a_v^-\) exists as \(\delta(G) \geq 8\chi(G)\). Set \(a_v^+ = a_v^- + 2\chi(G)\).

Furthermore, such a choice of \(a_v^-\) and \(a_v^+\) satisfy the conditions of Lemma 2.1, i.e.,

\[
\frac{1}{2}(d(v) - a_v^- - 2\chi(G)) - \chi(G) = \frac{1}{2}(d(v) - a_v^+) - \chi(G) \\
\geq \frac{d(v)}{8} - \chi(G),
\]

thus there is a subgraph \(H\) such that for all \(v, d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}\). Set \(w(e) = 2\) for \(e \in E(H)\) and \(w(e) = 1\) for \(e \in E(G) - E(H)\). If \(v \in V_i\), we have

\[
\sum_{v \sim e} w(e) = 2d_H(v) + d_{G-H}(v) = d_G(v) + d_H(v) \equiv 2i, 2i + 1 \pmod{2\chi(G)}.
\]

Thus adjacent vertices in different parts of \(\{V_1, \ldots, V_{\chi(G)}\}\) have different arities. As each \(V_i\) is an independent set, these weights form a vertex-coloring 2-edge-weighting of \(G\). \(\square\)

**Theorem 2.7.** Given a nice bipartite graph \(G = (U, W)\), if there exists a vertex \(v\) such that \(d_G(v) \notin \{d_G(x) \mid x \in N(v)\}\) and \(G - v - N(v)\) is connected, then \(G\) admits a vertex-coloring 2-edge-weighting.

**Proof.** If \(|U| \cdot |W|\) is even, byLemma 2.5, the result follows. So we may assume that both \(|U|\) and \(|W|\) are odd. Let \(v \in U\) satisfy \(d_G(v) \notin \{d_G(x) \mid x \in N(v)\}\). Since \(G - v - N(v)\) is connected and \(|U - v|\) is even, by Corollary 2.4, \(G - v - N(v)\) has a vertex-coloring 2-edge-weighting such that \(c(x)\) is odd for all \(x \in U - v\) and \(c(y)\) is even for all \(y \in W - N(v)\). Now we assign every edge of \(E[N(v), U]\) with weight 2. Clearly \(c(x)\) is odd for all \(x \in U - v\) and \(c(y)\) is even for all \(y \in W\). Note that \(c(v) = 2d_G(v)\) and \(c(u) = 2d_G(u)\) for all \(u \in N(v)\). Since \(d_G(u) \neq d_G(v)\), so \(c(v) \neq c(u)\) for all \(u \in N(v)\). Thus we obtain a vertex-coloring 2-edge-weighting of \(G\). \(\square\)

**Theorem 2.8.** Given a nice bipartite graph \(G = (U, W)\), if there exists a vertex \(v\) of degree \(\delta(G)\) such that \(d_G(v) \notin \{d_G(x) \mid x \in N(v)\}\) and \(G - v\) is connected, then \(G\) admits a vertex-coloring 2-edge-weighting.

**Proof.** If \(|U| \cdot |W|\) is even, by Lemma 2.5, the result follows. So we may assume that both \(|U|\) and \(|W|\) are odd. Let \(v \in U\) satisfy \(d_G(v) = \delta(G)\) and \(d_G(v) \notin \{d_G(x) \mid x \in N(v)\}\). Now we consider two cases.

Case 1. \(\delta(G)\) is even.

In this case, \(|(U - v) \cup N(v)|\) is even and \(|W - N(v)|\) is odd. By Corollary 2.4, \(G - v\) has a vertex-coloring 2-edge-weighting such that \(c(x)\) is odd for all \(x \in (U - v) \cup N(v)\) and \(c(y)\) is even for all \(y \in W - N(v)\). Assigning the edges incident to \(v\) with weight 1. Then \(c(w)\) is even for all \(w \in W\) and \(c(u)\) is odd for all \(u \in U - v\). Note that \(c(v) = \delta(G) < d_G(u) \leq c(u)\) for all \(u \in N(v)\), so we obtain a vertex-coloring 2-edge-weighting of \(G\).

Case 2. \(\delta(G)\) is odd.

In this case, \(|(U - v) \cup N(v)|\) is odd and \(|W - N(v)|\) is even. By Corollary 2.4, \(G - v\) has a vertex-coloring 2-edge-weighting such that \(c(x)\) is even for all \(x \in (U - v) \cup N(v)\) and \(c(y)\) is odd for all \(y \in W - N(v)\). Since \(d_G(v) \notin \{d_G(x) \mid x \in N(v)\}\), similar to Case 1, assigning the edges incident to \(v\) with weight 1 induces a vertex-coloring 2-edge-weighting of \(G\). \(\square\)
3. 3-connected bipartite graphs

In this section, we continue the research in this direction and prove that there exists a vertex-coloring 2-edge-weighting in every 3-connected bipartite graph. The following lemma is an important step in proving this result.

Lemma 3.1. Let $G$ be a 3-connected non-regular bipartite graph with bipartition $(U, W)$. Let $u \in U$ with $d(u) = \delta(G)$ and $t \leq \delta - 1$. Denote $N^\delta(u) = \{v \mid d(v) = \delta, v \in N_C(u)\} = \{u_1, \ldots, u_t\}$. Then there exist $e_1, \ldots, e_t$, where $e_i$ is incident to vertex $u_i$ in $G - u$ $(i = 1, \ldots, t)$, such that $G - u - \{e_1, \ldots, e_t\}$ is connected.

Proof. Let $C_1, \ldots, C_t$ be the components of $G - u - N^\delta(u)$. If we shrink each component $C_i$ to a vertex $v_i$, then we obtain a bipartite multi-graph $H = (X, Y)$ associated with $G - u$ as follows:

$X = \{u_1, \ldots, u_t\}, Y = \{c_1, \ldots, c_s\}$ and $|E_H(u_i, c_j)| = |E_G(u_i, c_j)|$ for $1 \leq i \leq t$ and $1 \leq j \leq s$.

Clearly, $d_H(u_i) = \delta - 1$ for every $u_i \in X$.

Claim. $H$ contains a connected spanning subgraph $T$ such that $d_T(v) \leq \delta - 2$ for every $v \in X$.

Suppose that the claim does not hold. Let $R$ be a connected induced subgraph of $H$ satisfying

(i) $R$ contains a connected spanning subgraph $M$ such that $d_M(v) \leq \delta - 2$ for every $v \in V(M) \cap X$;
(ii) $|V(R)|$ is maximum.

It is easy to see that $V(R) \neq \emptyset$ and $R \neq H$. Let $R = (A, B)$, where $A \subseteq X$ and $B \subseteq Y$. By the maximality of $R$, we have $d_R(v) \geq \delta - 2$ for every $v \in A$ and $E_R(B, X - A) = \emptyset$. Let $L = \{v \mid d_R(v) = \delta - 2, v \in A\}$. We see $|L| \geq 2$ since $G$ is 3-connected. Let $M^*$ be a connected spanning subgraph of $R$ such that $d_{R^*}(v) = \delta - 2$ for every $v \in A$. Note that for every connected spanning subgraph $N^*$ of $M^*$, we have $d_{N^*}(v) = \delta - 2$ for $v \in L$ by the maximality of $R$. So every edge incident with $w$ in $M^*$, where $w \in L$, is a cut-edge of $M^*$. Let $|L| = l$ and $|E(R) - E(M^*)| = m$. Then $l + m \leq t \leq \delta - 1$. We have

$$\omega(M^* - L) = \omega(H - L - (E(R) - E(M^*))) - 1 \geq (\delta - 3)l + 1.$$ 

So $\omega(H - L) \geq (\delta - 3)l + 2 - m$, which implies

$$\omega(G - u - L) \geq (\delta - 3)l + 2 - m + 1 \geq (\delta - 3)l + 2 - (\delta - 1 - l) = (\delta - 2)l + 3 - \delta.$$ 

Since $G$ is 3-connected, then

$$3\omega(G - u - L) \leq (\delta - 1)l + \delta - l.$$ 

It follows that

$$\omega(G - u - L) \leq \left\lfloor \frac{(\delta - 1)l + \delta - l}{3} \right\rfloor.$$ 

However

$$(\delta - 2)l + 3 - \delta - \frac{(\delta - 1)l + \delta - l}{3} = \frac{2\delta l}{3} - \frac{4\delta}{3} - \frac{4l}{3} + 3 > 0,$$

a contradiction. So we complete the claim and thus obtain a connected spanning subgraph $T$ of $H$.

Let $E'$ denote the set of corresponding edges of $E(T)$ in $G$. Then we obtain a spanning subgraph $T^* = \bigcup_{i=1}^{s} C_i \cup N^\delta(u) \cup E'$ of $G - u$ such that $d_{T^*}(v) \leq \delta - 2$ for every $v \in N^\delta(u)$. Thus the proof is complete. □

The following theorem is the main result of this section.

Theorem 3.2. Let $G = (U, W)$ be a nice bipartite graph. If $G$ is 3-connected, then $G$ admits a vertex-coloring 2-edge-weighting.
Proof. If $G$ is a regular graph, the result follows from Lemma 2.5(3). So we may assume that $G$ is a 3-connected non-regular bipartite graph with partition $(U, W)$. Let $u \in U$ with $d(u) = \delta(G)$ and $N^k(u) = \{v \mid d(v) = \delta, v \in N_c(u)\} = \{u_1, \ldots, u_t\}$, where $t \leq \delta - 1$. Then by Lemma 3.1, there exist $e_1, \ldots, e_t$ where $e_i$ is incident to vertex $u_i$ in $G - u$ for $i = 1, \ldots, t$, such that $G - u - \{e_1, \ldots, e_t\}$ is connected.

By Lemma 2.5, we can assume that $|U| \mid |W|$ is odd. Now we consider two cases.

Case 1. $\delta(G)$ is even.

Then $|N(u) \cup (U - u)|$ is even. By Corollary 2.4, $G - u - \{e_1, \ldots, e_t\}$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is odd for all $x \in N(u) \cup (U - u)$ and $c(y)$ is even for all $y \in W - N(u)$. We assign every edge of $\{e_1, \ldots, e_t\}$ with weight 2 and every edge of $\{uu_i \mid i = 1, \ldots, t\}$ with weight 1. Then $c(x)$ is odd for all $x \in U - u$, and $c(y)$ is even for all $y \in W$. Moreover, $c(u_i) > d(u_i) = d(u) = c(u)$ for all $i = 1, \ldots, t$. Note that $c(y) \geq d(y) > d(u) = c(u)$ for all $y \in N(u) - N^k(u)$. Hence we obtain a vertex-coloring 2-edge-weighting of $G$.

Case 2. $\delta(G)$ is odd.

Then $|W - N(u)|$ is odd. By Corollary 2.4, $G - u - \{e_1, \ldots, e_t\}$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is even for all $x \in N(u) \cup (U - u)$ and $c(y)$ is odd for all $y \in W - N(u)$. We assign every edge of $\{e_1, \ldots, e_t\}$ with weight 2 and every edge of $\{uu_i \mid i = 1, \ldots, t\}$ with weight 1. Similar to Case 1, $c(u_i) = d(u)$ and $c(u_i) < c(u_i)$ for $i = 1, \ldots, t$. Moreover, $c(u_i)$ is odd for $i = 1, \ldots, t$. Then we obtain a vertex-coloring 2-edge-weighting of $G$.

We complete the proof. □

Based on the proof of Theorem 3.2, we can easily obtain the following corollary.

**Corollary 3.3.** Let $G = (U, W)$ be a bipartite graph with $\delta(G) \geq 3$. If there exists a vertex of degree $\delta(G)$ such that $G - u - N(u)$ is connected, then $G$ admits a vertex-coloring 2-edge-weighting.

4. Conclusions

Karoński et al. [12] showed that for any fixed $p \in (0, 1)$ the random graph $G_{n,p}$ of order $n$ almost surely admits a vertex-coloring 2-edge-weighting. That is, the edges of almost all graphs can be labeled with 1 or 2 to induce a proper vertex-coloring. So a natural question is to classify all graphs which admit vertex-coloring 2-edge-weighting.

As an initial step towards this investigation, one may study bipartite graphs first. Since there exist families of infinite bipartite graphs (e.g., the generalized $\theta$-graphs) which only admit vertex-coloring 3-edge-weightings, it is nontrivial to classify all bipartite graphs with vertex-coloring 2-edge-weighting. In light of Theorem 3.2, it remains an open problem to classify all 2-connected bipartite graphs which admit a vertex-coloring 2-edge-weighting.

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